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Sharp Forelli–Rudin estimates and the norm of the Bergman projection ☆

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A R T I C L E I N F O

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A B S T R A C T

The purpose of this paper is twofold. We first establish a sharp version of Forelli–Rudin estimates for certain integrals on the ball. Then, as main application of these estimates, we obtain a sharp L^p-norm estimate for the Bergman projection, which refines a result of K. Zhu as well as gives a negative answer to a question raised by M. Dostanić.

1. Introduction

1.1. Bergman projection

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$, $n \geq 1$, and ν the Lebesgue volume measure on $\mathbb{B}_n$ normalized so that $\nu(\mathbb{B}_n) = 1$, while σ is the normalized surface measure on its

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boundary $\mathbb{S}_n$. As usual, for $p > 0$, the space $L^p(\mathbb{B}_n)$ consists of all Lebesgue measurable functions f on $\mathbb{B}_n$ for which

$$\|f\|_p := \left\{ \int_{\mathbb{B}_n} \left|f(z)\right|^p d\nu(z) \right\}^{\frac{1}{p}}$$

is finite. The Bergman space A^p consists of holomorphic functions f in $L^p(\mathbb{B}_n)$.

We consider the natural projection onto these spaces, i.e., the projection from $L^2(\mathbb{B}_n)$ onto A^2, also known as the Bergman projection. It can be expressed as an integral operator:

$$Pf(z) = \int_{\mathbb{B}_n} \frac{f(w)d\nu(w)}{(1 - \langle z, w\rangle)^{n+1}}.$$

Here, for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$,

$$\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n.$$

It is well known that, for $1 < p < \infty$, the Bergman projection P maps $L^p(\mathbb{B}_n)$ boundedly onto A^p. See, for example, [7] or Section 7.1 of [16]. A natural and interesting question is to determine the exact value of the L^p-operator norm $\|P\|_p$ of this operator. This turns out to be a difficult task to accomplish, except for the trivial case when $p = 2$. However, K. Zhu [19] obtained the following:

Theorem A. *There exists a constant $C > 0$, depending on n but not on p, such that*

$$C^{-1} \csc \frac{\pi}{p} \le \|P\|_p \le C \csc \frac{\pi}{p}$$

for all $1 < p < \infty$.

We remark that Zhu actually obtained the above result for the weighted Bergman projection and, for simplicity, we quote here only the unweighted version. Also, similar results were obtained in other settings, see [2,8,12].

A more demanding problem is to find explicit bounds for $\|P\|_p$. In the case when $n = 1$, Dostanić [4] proved the following.

Theorem B. *When $n = 1$, we have*

$$\frac{1}{2} \csc \frac{\pi}{p} \le \|P\|_p \le \pi \csc \frac{\pi}{p}$$

for all $1 < p < \infty$.