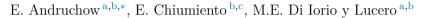


Contents lists available at ScienceDirect

## Journal of Functional Analysis

www.elsevier.com/locate/jfa

## Essentially commuting projections



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#### ARTICLE INFO

Article history: Received 30 May 2014 Accepted 1 October 2014 Available online 16 October 2014 Communicated by Alain Connes

MSC: 47A53 46L05 53C22

Keywords: Projections Compact operators Fredholm index Geodesics

#### ABSTRACT

Let  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$  be a fixed orthogonal decomposition of a Hilbert space, with both subspaces of infinite dimension, and let  $E_+, E_-$  be the projections onto  $\mathcal{H}_+$  and  $\mathcal{H}_-$ . We study the set  $\mathcal{P}_{cc}$  of orthogonal projections P in  $\mathcal{H}$  which essentially commute with  $E_+$  (or equivalently with  $E_-$ ), i.e.

 $[P, E_+] = PE_+ - E_+P \quad \text{is compact.}$ 

By means of the projection  $\pi$  onto the Calkin algebra, one sees that these projections  $P \in \mathcal{P}_{cc}$  fall into nine classes. Four discrete classes, which correspond to  $\pi(P)$  being 0, 1,  $\pi(E_+)$  or  $\pi(E_-)$ , and five essential classes which we describe below. The discrete classes are, respectively, the finite rank projections, finite co-rank projections, the Sato Grassmannian of  $\mathcal{H}_+$  and the Sato Grassmannian of  $\mathcal{H}_-$ . Thus the connected components of each of these classes are parametrized by the integers (via de rank, the co-rank or the Fredholm index, respectively). The essential classes are shown to be connected. We are interested in the geometric structure of  $\mathcal{P}_{cc}$ , being the set of selfadjoint projections of the  $C^*$ -algebra  $\mathcal{B}_{cc}$  of operators in  $\mathcal{B}(\mathcal{H})$  which essentially commute with  $E_+$ . In particular, we study the problem of existence of minimal geodesics joining two given projections in the same component. We show that

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Functional Analysis

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the Hopf–Rinow Theorem holds in the discrete classes, but not in the essential classes. Conditions for the existence and uniqueness of geodesics in these latter classes are found. © 2014 Elsevier Inc. All rights reserved.

### 1. Introduction

Let  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$  be a fixed decomposition of a separable Hilbert space, with both  $\mathcal{H}_+, \mathcal{H}_-$  infinite dimensional. Denote by  $E_+$  and  $E_-$  the orthogonal projections onto  $\mathcal{H}_+$  and  $\mathcal{H}_-$ , respectively. We shall study the unitary group  $\mathcal{U}_{cc}$  and the set of projections  $\mathcal{P}_{cc}$  of the  $C^*$ -algebra  $\mathcal{B}_{cc} = \mathcal{B}_{cc}(\mathcal{H}; \mathcal{H}_+, \mathcal{H}_-)$  given by

$$\mathcal{B}_{cc} = \{T \in \mathcal{B}(\mathcal{H}) : [T, E_+] \text{ is compact}\}.$$

Here [, ] denotes the commutator. Note that this condition is equivalent to  $[T, E_{-}]$  compact. If we denote by J the symmetry which is the identity in  $\mathcal{H}_{+}$  and minus the identity in  $\mathcal{H}_{-}$  (i.e.  $J = 2E_{+} - 1 = 1 - 2E_{-}$ ), this condition is equivalent to [T, J] compact. If one writes operators in  $\mathcal{H}$  as two by two matrices in terms of the given decomposition, elements in  $\mathcal{B}_{cc}$  have compact off-diagonal entries (with this matricial characterization, it is straightforward to verify that  $\mathcal{B}_{cc}$  is a  $C^*$ -algebra). If we denote by

$$\pi: \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$$

the homomorphism onto the Calkin algebra, and  $e_{+} = \pi(E_{+})$ , then

$$\mathcal{B}_{cc} = \pi^{-1} \big( \{ e_+ \}' \big),$$

where  $\{e_+\}'$  denotes the set of elements in  $\mathcal{C}(\mathcal{H})$  that commute with  $e_+$ .

The set  $\mathcal{P}_{cc}$  relates to the so called restricted or Sato Grassmannian (see e.g. [13,14], or [5,12] for a version using Hilbert–Schmidt operators instead of compact operators). In fact,  $\mathcal{P}_{cc}$  is disconnected, and several of its components form the restricted Grassmannian of  $\mathcal{H}_+$  (as well as the restricted Grassmannian of  $\mathcal{H}_-$ ). Thus this framework enables one to regard the restricted Grassmannian as (certain components of) the set of projections of a  $C^*$ -algebra. Again, by means of the homomorphism  $\pi$ , one sees that  $\mathcal{P}_{cc}$  decomposes into nine classes. If  $P \in \mathcal{P}_{cc}$ , then  $\pi(P)$  is one of the following (written as  $2 \times 2$  matrices in terms of  $e_+, e_- = 1 - e_+$ ):

$$0, 1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p_+ & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & p_- \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \text{ and } \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}.$$

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