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Essentially commuting projections

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ABSTRACT

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of a Hilbert space, with both subspaces of infinite dimension, and let E_+, E_- be the projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$. We study the set $\mathcal{P}_{cc}$ of orthogonal projections P in $\mathcal{H}$ which *essentially commute* with E_+ (or equivalently with E_-), i.e.

$$[P, E_+] = PE_+ - E_+P \quad \text{is compact.}$$

By means of the projection π onto the Calkin algebra, one sees that these projections $P \in \mathcal{P}_{cc}$ fall into nine classes. Four *discrete* classes, which correspond to $\pi(P)$ being 0, 1, $\pi(E_+)$ or $\pi(E_-)$, and five *essential* classes which we describe below. The discrete classes are, respectively, the finite rank projections, finite co-rank projections, the Sato Grassmannian of $\mathcal{H}_+$ and the Sato Grassmannian of $\mathcal{H}_-$. Thus the connected components of each of these classes are parametrized by the integers (via de rank, the co-rank or the Fredholm index, respectively). The essential classes are shown to be connected. We are interested in the geometric structure of $\mathcal{P}_{cc}$, being the set of selfadjoint projections of the C^*-algebra $\mathcal{B}_{cc}$ of operators in $\mathcal{B}(\mathcal{H})$ which essentially commute with E_+. In particular, we study the problem of existence of minimal geodesics joining two given projections in the same component. We show that

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the Hopf–Rinow Theorem holds in the discrete classes, but not in the essential classes. Conditions for the existence and uniqueness of geodesics in these latter classes are found.

1. Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed decomposition of a separable Hilbert space, with both $\mathcal{H}_+, \mathcal{H}_-$ infinite dimensional. Denote by E_+ and E_- the orthogonal projections onto $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. We shall study the unitary group $\mathcal{U}_{cc}$ and the set of projections $\mathcal{P}_{cc}$ of the C^*-algebra $\mathcal{B}_{cc} = \mathcal{B}_{cc}(\mathcal{H}; \mathcal{H}_+, \mathcal{H}_-)$ given by

$$\mathcal{B}_{cc} = \{T \in \mathcal{B}(\mathcal{H}) : [T, E_+] \text{ is compact}\}.$$

Here [,] denotes the commutator. Note that this condition is equivalent to $[T, E_-]$ compact. If we denote by J the symmetry which is the identity in $\mathcal{H}_+$ and minus the identity in $\mathcal{H}_-$ (i.e. $J = 2E_+ - 1 = 1 - 2E_-$), this condition is equivalent to $[T, J]$ compact. If one writes operators in $\mathcal{H}$ as two by two matrices in terms of the given decomposition, elements in $\mathcal{B}_{cc}$ have compact off-diagonal entries (with this matricial characterization, it is straightforward to verify that $\mathcal{B}_{cc}$ is a C^*-algebra). If we denote by

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

the homomorphism onto the Calkin algebra, and $e_+ = \pi(E_+)$, then

$$\mathcal{B}_{cc} = \pi^{-1}(\{e_+\}'),$$

where $\{e_+\}'$ denotes the set of elements in $\mathcal{C}(\mathcal{H})$ that commute with e_+.

The set $\mathcal{P}_{cc}$ relates to the so called restricted or Sato Grassmannian (see e.g. [13,14], or [5,12] for a version using Hilbert–Schmidt operators instead of compact operators). In fact, $\mathcal{P}_{cc}$ is disconnected, and several of its components form the restricted Grassmannian of $\mathcal{H}_+$ (as well as the restricted Grassmannian of $\mathcal{H}_-$). Thus this framework enables one to regard the restricted Grassmannian as (certain components of) the set of projections of a C^*-algebra. Again, by means of the homomorphism π, one sees that $\mathcal{P}_{cc}$ decomposes into nine classes. If $P \in \mathcal{P}_{cc}$, then $\pi(P)$ is one of the following (written as 2×2 matrices in terms of $e_+, e_- = 1 - e_+$):

$$0,\ 1,\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} p_+ & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 0 \\ 0 & p_- \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & p_- \end{pmatrix} \text{ and } \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}.$$