Journal of Functional Analysis 268 (2015) 376–416

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Weighted variation inequalities for differential operators and singular integrals

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A R T I C L E   I N F O

Article history:
Received 10 May 2014
Accepted 7 October 2014
Available online 18 October 2014
Communicated by L. Saloff-Coste

MSC:
primary 42B20, 42B25
secondary 46E30

Keywords:
Variation inequalities
Vector-valued inequalities
(One-sided) A_p weights
Differential operators
Singular integrals

A B S T R A C T

We prove weighted strong q-variation inequalities with $2 < q < \infty$ for differential and singular integral operators. For the first family of operators the weights used can be either Sawyer's one-sided A_p^+ weights or Muckenhoupt's A_p weights according to whether the differential operators in consideration are one-sided or symmetric. We use only Muckenhoupt's A_p weights for the second family. All these inequalities hold equally in the vector-valued case, that is, for functions with values in ℓ^ρ for $1 < \rho < \infty$. As application, we show variation inequalities for mean bounded positive invertible operators on L^p with positive inverses.

1. Introduction

Variation inequalities have been the subject of many recent research papers in probability, ergodic theory and harmonic analysis. One important feature of these inequalities

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http://dx.doi.org/10.1016/j.jfa.2014.10.008

is the fact that they immediately imply the pointwise convergence of the underlying family of operators without using the Banach principle via the corresponding maximal inequality. Moreover, these variation inequalities can be used to measure the speed of convergence of the family.

The first variation inequality was proved by Lépingle [19] for martingales which improves the classical Doob maximal inequality (see also [27] for a different approach and related results). Thirteen years later, Bourgain [1] proved the variation inequality for the ergodic averages of a dynamic system. Bourgain's work has inaugurated a new research direction in ergodic theory and harmonic analysis. It was considerably improved by subsequent works and largely extended to many other operators in ergodic theory; see, for instance, [14,13,18]. Almost in the same period, variation inequalities have been studied in harmonic analysis too. The first work on this subject is [2] in which Campbell, Jones, Reinhold and Wierdl proved the variation inequalities for the Hilbert transform. Since then many other publications came to enrich the literature on this subject (cf. e.g. [3,6,9,12,15,16,23–26]).

The purpose of this paper is to study weighted variation inequalities for differential and singular integral operators. The first family of operators can be considered both in the discrete and continuous cases. To fix ideas let us confine ourselves to the former. Given a function f on $\mathbb{Z}$ define

$$A_N^+(f)(n) = \frac{1}{N+1}\sum_{i=0}^{N} f(n+i)$$

and $\mathcal{A}^+(f)(n) = \{A_N^+(f)(n)\}_{N\geq 0}$. $\mathcal{A}^+$ is an operator mapping functions on $\mathbb{Z}$ to sequences of functions on $\mathbb{Z}$. We will study the variation of the sequence $\mathcal{A}^+(f)(n)$.

Let $1 \leq q < \infty$ and $a = \{a_N\}_{N\geq 0}$ be a sequence of complex numbers. The q-variation of a is defined as

$$\|a\|_{v_q} = \sup\left(\sum_{j=0}^{\infty} |a_{N_j} - a_{N_{j+1}}|^q\right)^{1/q}, \tag{1.1}$$

where the supremum runs over all increasing sequences $\{N_j\}$ of nonnegative integers. Let v_q denote the space of all sequences with finite q-variation. This is a Banach space modulo constant functions. Let $\mathcal{V}_q\mathcal{A}^+(f)(n) = \|\mathcal{A}^+(f)(n)\|_{v_q}$. Thus the operator $\mathcal{V}_q\mathcal{A}^+$ sends functions on $\mathbb{Z}$ to nonnegative functions on $\mathbb{Z}$. Throughout the paper, $\mathcal{V}_q$ designates the operator which maps a sequence to its q-variation. Later in the continuous case, the same symbol $\mathcal{V}_q$ will also be the operator mapping functions on $(0,\infty)$ to their q-variations.

Bourgain's theorem quoted before asserts that for any $2 < q < \infty$, $\mathcal{V}_q\mathcal{A}^+$ is bounded on $\ell^2(\mathbb{Z})$. This result was extended to $\ell^p(\mathbb{Z})$ for any $1 < p < \infty$ in [13]. Moreover, Jones et al. also proved that $\mathcal{V}_q\mathcal{A}^+$ is of weak type $(1,1)$, namely, it maps $\ell^1(\mathbb{Z})$ into $\ell^{1,\infty}(\mathbb{Z})$.