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Finite groups acting on higher dimensional noncommutative tori

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ABSTRACT

For the canonical action α of $\mathrm{SL}_2(\mathbb{Z})$ on 2-dimensional simple rotation algebras $\mathcal{A}_\theta$, it is known that if F is a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the crossed products $\mathcal{A}_\theta \rtimes_\alpha F$ are all AF algebras. In this paper we show that this is not the case for higher dimensional noncommutative tori. More precisely, we show that for each $n \geq 3$ there exist noncommutative simple $\phi(n)$-dimensional tori $\mathcal{A}_\Theta$ which admit canonical action of $\mathbb{Z}_n$ and for each odd $n \geq 7$ with $2\phi(n) \geq n + 5$ their crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ are not AF (with nonzero K_1-groups). It is also shown that the only possible canonical action by a finite group on a 3-dimensional simple torus is the flip action by $\mathbb{Z}_2$. Besides, we discuss the canonical actions by finite groups $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}$, and $\mathbb{Z}_{12}$ on the 4-dimensional torus of the form $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$.

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1. Introduction

The *rotation algebra* $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is the universal C^*-algebra generated by two unitaries u_1, u_2 satisfying the commutation relation $u_2u_1 = \exp(2\pi i\theta)u_1u_2$. If u_1 and u_2 commute (that is, if $\theta \in \mathbb{Z}$), $\mathcal{A}_\theta$ is isomorphic to the commutative C^*-algebra $C(\mathbb{T}^2)$ of all continuous functions on the 2-dimensional torus $\mathbb{T}^2$, and so the rotation algebras $\mathcal{A}_\theta$ are often called 2-dimensional noncommutative tori. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{A}_\theta$ is called an *irrational rotational algebra* and this is the case exactly when $\mathcal{A}_\theta$ is a simple C^*-algebra.

More generally, for $d \geq 2$, a *noncommutative d-dimensional torus* (or simply a *d-torus*) $\mathcal{A}_\Theta$ associated with a skew symmetric real $d \times d$ matrix $\Theta = (\theta_{kj})$ is the universal C^*-algebra generated by d unitaries $u_1, \ldots, u_d$ that are subject to the commutation relations

$$u_ju_k = \exp(2\pi i\theta_{kj})u_ku_j. \tag{1.1}$$

$\mathcal{A}_\Theta$ was introduced in [11] as the twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ of $\mathbb{Z}^d$ twisted by the 2-cocycle ω_Θ given by $\omega_\Theta(x, y) = \exp(\pi i\langle\Theta x, y\rangle)$ for $x, y \in \mathbb{Z}^d$.

In [16] Watatani considered an automorphism α_A, $A = (a_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$, on an irrational rotational algebra $\mathcal{A}_\theta$ defined by

$$\alpha_A(u_1) = \exp(\pi i\theta a_{11}a_{21})u_1^{a_{11}}u_2^{a_{21}}, \qquad \alpha_A(u_2) = \exp(\pi i\theta a_{12}a_{22})u_1^{a_{12}}u_2^{a_{22}} \tag{1.2}$$

and then classified these automorphisms using the notion of K_1-entropy. Brenken [1] used the automorphism to study representations of rotational algebras. In this paper, the action $A \mapsto \alpha_A : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{A}_\theta)$ and its d-dimensional version (Definition 2.5) will be called a *canonical action*.

The group $\mathrm{SL}_2(\mathbb{Z})$ is known to have only four (up to conjugacy) nontrivial finite subgroups which are isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$. The crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_k$ of a simple $\mathcal{A}_\theta$ by the restriction of the canonical action α to $\mathbb{Z}_k$, $k = 2, 3, 4, 6$, are all known to be AF-algebras and moreover their K_0 groups are computed (see [3, Theorem 0.1]), which implies $\mathcal{A}_{\theta_1} \rtimes_\alpha \mathbb{Z}_k \cong \mathcal{A}_{\theta_2} \rtimes_\alpha \mathbb{Z}_l$ if and only if $k = l$ and $\theta_1 = \pm\theta_2 \bmod \mathbb{Z}$. Also it is known in the same paper [3] that $\mathcal{A}_\Theta \rtimes_\sigma \mathbb{Z}_2$ is an AF algebra if $\mathcal{A}_\Theta$ is a simple d-dimensional noncommutative torus and σ is the action given by the flip automorphism sending the unitary generators u_j to their adjoints u_j^* for $j = 1, \ldots, d$. This seminal work [3] was actually motivated, as reviewed in the first chapter there, by several previous studies including, for example, the result [15] that for most irrational numbers θ, the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_4$ are AF algebras, and it finally settled down the case $\mathcal{A}_\theta \rtimes_\alpha F$ for any (2-dimensional) irrational rotational algebras $\mathcal{A}_\theta$ and any finite groups $F \subset \mathrm{SL}_2(\mathbb{Z})$.

It would then be a very natural question to ask whether the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ of a simple higher dimensional noncommutative d-torus $\mathcal{A}_\Theta$ is still AF even when α is the canonical action of a finite subgroup G of $\mathrm{SL}_d(\mathbb{Z})$ (or $\mathrm{GL}_d(\mathbb{Z})$). But it was unclear, at least to the knowledge of the authors, even whether there are any known finite groups