Journal of Functional Analysis 268 (2015) 1053–1077

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Periodic solutions of the Degasperis–Procesi equation: Well-posedness and asymptotics ☆

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A R T I C L E I N F O

Article history:
Received 22 January 2014
Accepted 14 November 2014
Available online 25 November 2014
Communicated by C. De Lellis

MSC:
primary 35L65, 35G25, 35B10, 35B40
secondary 35L05, 35A05, 35B05

Keywords:
Shallow water equation
Hyperbolic conservation law
Entropy condition
Long-time asymptotics of periodic solutions

A B S T R A C T

We prove the well-posedness of periodic entropy (discontinuous) solutions for the Degasperis–Procesi equation. Partly motivated by the bounded periodic solutions found by Vakhnenko and Parkes [21], we study the long-time asymptotic behavior of periodic entropy solutions.

☆ This work was initiated while G.M.C. visited Department of Mathematics at the University of Oslo. He is grateful for Department of Mathematics's financial support and excellent working environment.

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http://dx.doi.org/10.1016/j.jfa.2014.11.008
0022-1236/

1. Introduction

We investigate the well-posedness and long-time asymptotic behavior of periodic discontinuous solutions of the *Degasperis–Procesi* equation. It has the form

$$\partial_t u - \partial_{txx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{1.1}$$

and is augmented with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

We assume

$$u_0 \in L^\infty(\mathbb{R}), \quad u_0 \text{ is 1-periodic.} \tag{1.3}$$

Degasperis and Procesi [11] deduced (1.1) from the following family of third order dispersive nonlinear equations, indexed over six constants $\alpha, \gamma, c_0, c_1, c_2, c_3 \in \mathbb{R}$:

$$\partial_t u + c_0 \partial_x u + \gamma \partial_{xxx}^3 u - \alpha^2 \partial_{txx}^3 u = \partial_x \left(c_1 u^2 + c_2 (\partial_x u)^2 + c_3 u \partial_{xx}^2 u\right).$$

Using the method of asymptotic integrability, they found that only three equations within this family were asymptotically integrable up to the third order: the *KdV equation* ($\alpha = c_2 = c_3 = 0$), the *Camassa–Holm equation* ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$), which properly scaled reads

$$\partial_t u + \partial_x u + 6u\partial_x u + \partial_{xxx}^3 u - \alpha^2 \left(\partial_{txx}^3 u + \frac{9}{2}\partial_x u \partial_{xx}^2 u + \frac{3}{2} u \partial_{xxx}^3 u\right) = 0. \tag{1.4}$$

One can transform (1.4) into the form (1.1), see [9,10] for details.

Degasperis, Holm and Hone [10] proved the integrability of (1.1) by constructing a Lax pair. Moreover, they provided a relation to a negative flow in the Kaup–Kupershmidt hierarchy by a reciprocal transformation and derived two infinite sequences of conserved quantities along with a bi-Hamiltonian structure. Furthermore, they showed that the Degasperis–Procesi equation is endowed with weak (continuous) solutions that are superpositions of multipeakons and described the (finite-dimensional) integrable peakon dynamics. An explicit solution was also found in the perfectly anti-symmetric peakon–antipeakon collision case. Lundmark and Szmigielski [15], using an inverse scattering approach, computed n-peakon solutions to (1.1). Mustafa [17] proved that smooth solutions have infinite speed of propagation. Blow-up phenomena have been investigated for example in [24]. Regarding the Cauchy problem and the initial-boundary value problem for the Degasperis–Procesi equation (1.1), Escher, Liu and Yin have studied its well-posedness within certain functional classes in a series of papers [12,22,23].