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Semi-Dirichlet forms, Feynman–Kac functionals and the Cauchy problem for semilinear parabolic equations

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ABSTRACT

In the first part of the paper we prove various results on regularity of Feynman–Kac functionals of Hunt processes associated with time-dependent semi-Dirichlet forms. In the second part we study the Cauchy problem for semilinear parabolic equations with measure data involving operators associated with time-dependent forms. Model examples are non-symmetric divergence form operators and fractional laplacians with possibly variable exponents. We first introduce a definition of a solution resembling Stampacchia's definition in the sense of duality and then, using the results of the first part, we prove the existence, uniqueness and regularity of solutions of the problem under mild assumptions on the data.

1. Introduction

Let E be a locally compact separable metric space, m be an everywhere dense Borel measure on E and let $\{B^{(t)}; t \in \mathbb{R}\}$ be a family of regular semi-Dirichlet forms on $L^2(E;m)$ with common domain F. Let us consider a time-dependent semi-Dirichlet form

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$$\mathcal{E}(u,v)=\begin{cases}(-\frac{\partial u}{\partial t},v)+\mathcal{B}(u,v), & (u,v)\in\mathcal{W}\times L^2(\mathbb{R};F),\\ (u,\frac{\partial v}{\partial t})+\mathcal{B}(u,v), & (u,v)\in L^2(\mathbb{R};F)\times\mathcal{W},\end{cases}$$

where $\mathcal{W}=\{u\in L^2(\mathbb{R};F);\ \frac{\partial u}{\partial t}\in L^2(\mathbb{R};F')\}$, $(\cdot,\cdot)$ stands for the duality pairing between $L^2(\mathbb{R};F)$ and $L^2(\mathbb{R};F')$, and

$$\mathcal{B}(u,v)=\int_{\mathbb{R}}B^{(t)}\big(u(t),v(t)\big)\,dt.$$

Let $\mathbb{M}=(\{\mathbf{X}_t,\ t\geq 0\},\{P_z,\ z\in\mathbb{R}\times E\})$ be a Hunt process with life-time ζ associated with $\mathcal{E}$. The main object of the present paper is to study regularity of the Feynman–Kac functionals of the form

$$u(z)=E_z\varphi(\mathbf{X}_{\zeta_\tau})+E_z\int_0^{\zeta_\tau}dA^\mu_r,\quad z\in E_{0,T}\equiv(0,T]\times E. \tag{1.1}$$

Here E_z denotes the expectation with respect to P_z, $\zeta_\tau=\zeta\wedge(T-\tau(0))$, where τ is the uniform motion to the right, $\varphi:E\to\mathbb{R}$ and A^μ is the unique natural additive functional of $\mathbb{M}$ in Revuz correspondence with a smooth measure μ on $E_{0,T}$ (see Section 2).

Our interest in functionals of the form (1.1) comes from the fact that regularity of u implies regularity of solutions of the Cauchy problem

$$-\frac{\partial u}{\partial t}-L_tu=\mu,\qquad u(T)=\varphi, \tag{1.2}$$

where L_t is the operator associated with the form $B^{(t)}$. The study of equations of the form (1.2) and more general semilinear equations of the form

$$-\frac{\partial u}{\partial t}-L_tu=f(t,x,u)+\mu,\qquad u(T)=\varphi \tag{1.3}$$

is the second main goal of the paper. We are interested in equations with $\varphi\in L^1(E;m)$ and "true" measure data. In the paper we assume that μ belongs to the space $\mathcal{R}(E_{0,T})$ of all smooth (with respect to the capacity determined by $\mathcal{E}$) measures on $E_{0,T}$ such that $E_zA^{|\mu|}_{\zeta_\tau}<\infty$ for $\mathcal{E}$-quasi-every (q.e.) $z\in E_{0,T}$, and that $\delta_{\{T\}}\otimes\varphi\cdot m\in\mathcal{R}(E_{0,T})$. These are minimal assumptions on μ, φ under which u is finite m_1-a.e., where $m_1=dt\otimes m$, and hence finite $\mathcal{E}$-q.e. The class $\mathcal{R}(E_{0,T})$ is quite wide. If $\mathcal{E}$ satisfies some duality condition (see condition (Δ) below) then it includes the space $\mathcal{M}_{0,b}(E_{0,T})$ of all bounded smooth measures on $E_{0,T}$. Our general framework of time-dependent semi-Dirichlet forms associated with the family of semi-Dirichlet forms allows us to study (1.2), (1.3) for wide class of local and nonlocal operators L_t. Model examples are diffusion operators with drift terms and fractional laplacians with constant and variable exponents (for more examples see [11,15,16,21,24]). We think that applicability of our general results