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On the ground state energy of the Laplacian with a magnetic field created by a rectilinear current

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We consider the three-dimensional Laplacian with a magnetic field created by an infinite rectilinear current bearing a constant current. The spectrum of the associated Hamiltonian is the positive half-axis as the range of an infinity of band functions all decreasing toward 0. We make a precise asymptotics of the band functions near the ground state energy and we exhibit a semi-classical behavior. We perturb the Hamiltonian by an electric potential. Helped by the analysis of the band functions, we show that for slow decaying potential an infinite number of negative eigenvalues are created whereas only finite number of eigenvalues appear for fast decaying potential. The criterion about finiteness depends essentially on the decay rate of the potential with respect to the distance to the wire.

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1. Introduction

1.1. Motivation and problematic

Physical context We consider in $\mathbb{R}^3$ the magnetic field created by an infinite rectilinear wire bearing a constant current. Let (x, y, z) be the cartesian coordinates of $\mathbb{R}^3$ and assume that the wire coincides with the z-axis. Due to the Biot and Savard law, the generated magnetic field writes

$$\mathbf{B}(x,y,z) = \frac{1}{r^2}(-y, x, 0)$$

where $r := \sqrt{x^2+y^2}$ is the radial distance corresponding to the distance to the wire. Let $\mathbf{A}(x,y,z) := (0, 0, \log r)$ be a magnetic potential satisfying $\operatorname{curl} \mathbf{A} = \mathbf{B}$. We define the unperturbed magnetic Hamiltonian

$$H_{\mathbf{A}} := (-i\nabla - \mathbf{A})^2 = D_x^2 + D_y^2 + (D_z - \log r)^2; \qquad D_j := -i\partial_j$$

initially defined on $C_0^\infty(\mathbb{R}^3)$ and then closed in $L^2(\mathbb{R}^3)$. It is known (see [22], and [23] for a more general setting) that the spectrum of $H_{\mathbf{A}}$ has a band structure with band functions defined on $\mathbb{R}$ and decreasing from $+\infty$ toward 0. Then the spectrum of $H_{\mathbf{A}}$ is absolutely continuous and coincides with $[0, +\infty)$. In that case the presence of the magnetic field does not change the spectrum (i.e. $\mathfrak{S}(H_{\mathbf{A}}) = \mathfrak{S}(-\Delta)$), that may be expected since the magnetic field tends to 0 far from the wire. In this article we study the ground state energy of $H_{\mathbf{A}}$ and its stability under electric perturbation. These questions are related to the dynamic of spinless quantum particles submitted to the magnetic field $\mathbf{B}$ and perturbed by an electric potential.

Comparison with the free Hamiltonian In general the spectrum of a Laplacian may be higher in the presence of a magnetic field (see [2]). As already said, in our model we still have $\mathfrak{S}(H_{\mathbf{A}}) = \mathbb{R}_+$. However the dynamics are very different from the free motion, see [22] for a description of the classical and quantum dynamics of this model. As we will see, the behavior of the negative spectrum under electrical perturbation is also different than what happens without magnetic field.

If V is a multiplication operator by a real electric potential V such that $V(H_{\mathbf{A}}+1)^{-1}$ is compact then the operator $H_{\mathbf{A}} - V$ is self-adjoint, its essential spectrum coincides with the positive half-axis and discrete spectrum may appear under 0.

Let us recall that, due to the diamagnetic inequality (see [2, Section 2]), the operator $V(H_{\mathbf{A}}+1)^{-1}$ is compact as soon as $V(-\Delta+1)^{-1}$ is compact. For any self-adjoint operator H, we denote by $\mathcal{N}(H, \lambda)$ the number of eigenvalues of H below $-\lambda < 0$. Then we have ([2, Theorem 2.15]):

$$\mathcal{N}(H_{\mathbf{A}} - V, 0^+) \leqslant C \int_{\mathbb{R}^3} V_+(x,y,z)^{\frac{3}{2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad V_+ := \max(0, V). \tag{1.1}$$