Journal of Functional Analysis 267 (2014) 1889–1916

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Nonuniform dichotomy spectrum and normal forms for nonautonomous differential systems

Xiang Zhang [1]

Department of Mathematics, and MOE–LSC, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

ARTICLE INFO

Article history:
Received 2 January 2011
Accepted 29 July 2014
Available online 8 August 2014
Communicated by I. Rodnianski

MSC:
34A34
34C41
37G05
58D05

Keywords:
Nonuniform exponential dichotomy
Nonuniform dichotomy spectrum
Nonautonomous differential system
Normal form

ABSTRACT

The aim of this paper is to study the normal forms of nonautonomous differential systems. For doing so, we first investigate the nonuniform dichotomy spectrum of the linear evolution operators that admit a nonuniform exponential dichotomy, where the linear evolution operators are defined by nonautonomous differential equations $\dot{x} = A(t)x$ in $\mathbb{R}^n$. Using the nonuniform dichotomy spectrum we obtain the normal forms of the nonautonomous linear differential equations. Finally we establish the finite jet normal forms of the nonlinear differential systems $\dot{x} = A(t)x + f(t, x)$ in $\mathbb{R}^n$, which is based on the nonuniform dichotomy spectrum and the normal forms of the nonautonomous linear systems.

1. Introduction and statement of the main results

The normal form theory in dynamical systems is to simplify ordinary differential equations through the change of variables. This theory can be traced back to Poincaré [21]. Some classical results in this direction for autonomous differential systems

E-mail address: xzhang@sjtu.edu.cn.

[1] The author is partially supported by NNSF of China grant 11271252, RFDP of Higher Education of China grant 20110073110054, and FP7-PEOPLE-2012-IRSES-316338 of Europe.

http://dx.doi.org/10.1016/j.jfa.2014.07.029
0022-1236/

are the Poincaré–Dulac normal form theorem [22], Siegel's theorem [23], Hartman–Grobman's theorem [11,12], Sternberg's theorem [27,28], Chen's theorem [7], Takens' theorem [31] and so on. See also [6,8,13,14,33] and the recent survey paper [29] and the references therein. For nonautonomous systems, Barreira and Valls had established several results on the topological conjugacy between nonuniformly hyperbolic dynamical systems (see e.g. [2–5]). Using the resonance of the dichotomy spectrum to study the normal forms of nonautonomous system, Siegmund [26] obtained a finite order normal form, and Wu and Li [32] got analytic normal forms of a class of analytic nonautonomous differential systems. As our knowledge, these last two papers are the only ones in which the normal forms of nonautonomous systems via the dichotomy spectrums were studied. Recently Li, Llibre and Wu [15] and [16] also had studied the normal forms of almost periodic differential and difference equations, respectively. For random differential systems there also appeared some results on normal forms [18–20], in which they extended Poincaré's, Sternberg's and Siegel's normal form theorems for autonomous differential systems to random dynamical systems.

As well-known, the normal form theory has played important roles in the study of bifurcation and some related topics of dynamical systems. Recently this theory has been successfully applied to study the embedding flow problem of diffeomorphisms, see for instance [17,34–36].

In this paper we will study the normal forms of nonautonomous differential systems with their linear parts admitting a nonuniform exponential dichotomy. For this aim we first consider the nonautonomous linear differential systems in $\mathbb{R}^n$

$$\dot{x} = A(t)x, \tag{1.1}$$

with $A(t) \in M_n(\mathbb{R})$ the set of square matrix functions of nth order defined in $\mathbb{R}$, we assume in this paper that each solution of system (1.1) is defined on the whole $\mathbb{R}$. Denote by $\Phi(t,s)$ the evolution operator associated to system (1.1). Then we have

$$x(t) = \Phi(t,s)x(s), \qquad \Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau) \quad \text{for all } t,s,\tau \in \mathbb{R},$$

where $x(t)$ is a solution of system (1.1).

We say that system (1.1) admits a *nonuniform exponential dichotomy* if there exists an invariant projection $P(t) \in M_n(\mathbb{R})$ (where *invariant* means that $P(t)\Phi(t,s) = \Phi(t,s)P(s)$ for all $t,s \in \mathbb{R}$), and $K \geq 1$, $\alpha < 0 < \beta$ and $\mu,\nu \geq 0$ with $\alpha + \mu < 0$, $\beta - \nu > 0$ and $\max\{\mu,\nu\} \leq \min\{-\alpha,\beta\}$ such that

$$\begin{aligned} \big\|\Phi(t,s)P(s)\big\| &\leq Ke^{\alpha(t-s)+\mu|s|} \quad \text{for } t \geq s, \\ \big\|\Phi(t,s)\big(I - P(s)\big)\big\| &\leq Ke^{\beta(t-s)+\nu|s|} \quad \text{for } t \leq s. \end{aligned}$$