Journal of Functional Analysis 267 (2014) 2152–2198

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Parabolic Harnack inequality of viscosity solutions on Riemannian manifolds

Soojung Kim [a,*], Ki-Ahm Lee [b,c]

[a] *National Institute for Mathematical Sciences, 70 Yuseong-daero, 1689 Beon-gil, Yuseong-gu, Daejeon 306-390, Republic of Korea*
[b] *School of Mathematical Sciences, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 151-747, Republic of Korea*
[c] *Center for Mathematical Challenges, Korea Institute for Advanced Study, Seoul 130-722, Republic of Korea*

A R T I C L E   I N F O

Article history:
Received 5 May 2013
Accepted 25 July 2014
Available online 14 August 2014
Communicated by A. Borodin

Keywords:
Parabolic Harnack inequality on Riemannian manifolds
Viscosity solutions
Weak Harnack inequality
ABP–Krylov–Tso type estimate

A B S T R A C T

We consider viscosity solutions to nonlinear uniformly parabolic equations in nondivergence form on a Riemannian manifold M with the sectional curvature bounded from below by $-\kappa$ for $\kappa \geq 0$. In the elliptic case, Wang and Zhang [24] recently extended the results of [5] to nonlinear elliptic equations in nondivergence form on such M, where they obtained the Harnack inequality for classical solutions. We establish the Harnack inequality for nonnegative *viscosity solutions* to nonlinear uniformly *parabolic equations* in nondivergence form on M. The Harnack inequality of nonnegative viscosity solutions to the elliptic equations is also proved.

Contents

* Corresponding author.
E-mail addresses: soojung26@gmail.com (S. Kim), kiahm@math.snu.ac.kr (K.-A. Lee).

http://dx.doi.org/10.1016/j.jfa.2014.07.023
0022-1236/

1. Introduction and main results

In this paper, we study the Harnack inequality of viscosity solutions to nonlinear uniformly parabolic equations in nondivergence form on Riemannian manifolds. Let (M, g) be a smooth, complete Riemannian manifold of dimension n. Consider a nonlinear uniformly parabolic equation

$$F\left(D^2 u\right) - \partial_t u = f \quad \text{in } M \times \mathbb{R}, \tag{1}$$

where $D^2 u$ denotes the Hessian of the function u defined by

$$D^2 u(X, Y) = g(\nabla_X \nabla u, Y),$$

for any vector fields X, Y on M, and ∇u is the gradient of u. We notice that in the case, when F is the trace operator, (1) is the well-known heat equation with a source term.

In the setting of elliptic equations on M, Cabré [5] established the Krylov–Safonov type Harnack inequality of classical solutions to linear, uniformly elliptic equations in nondivergence form, when M has nonnegative sectional curvature. The Krylov–Safonov Harnack inequality is based on the Aleksandrov–Bakelman–Pucci (ABP) estimate, which is proved using affine functions in the Euclidean case. Since affine functions cannot be generalized into an intrinsic notion on Riemannian manifolds, Cabré considered the functions of the squared distance instead of the affine functions to overcome the difficulty. Later, Kim [14] improved Cabré's result removing the sectional curvature assumption and imposing the certain condition on the distance function (see [14, p. 283]). Recently, Wang and Zhang [24] obtained a version of the ABP estimate on M with Ricci curvature bounded from below, and the Harnack inequality of classical solutions for nonlinear uniformly elliptic operators provided that M has a lower bound of the sectional curvature.

In the parabolic case, the Krylov–Safonov Harnack inequality was proved in [15] for classical solutions to linear, uniformly parabolic equations in nondivergence form, assuming essentially the same condition introduced by Kim [14]. The result in [15], in particular, gives a non-divergent proof of Li–Yau's Harnack inequality for the heat equation in a manifold with nonnegative Ricci curvature [19]. The ABP–Krylov–Tso estimate discovered by Krylov [16] in the Euclidean case (see also [21,23]) is a parabolic analogue of the ABP estimate, and a key ingredient in proving the parabolic Harnack inequality. In order to prove the ABP–Krylov–Tso type estimate on Riemannian manifolds, an intrinsically geometric version of the Krylov–Tso normal map, namely,