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Stability of Pólya–Szegő inequality for log-concave functions

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ABSTRACT

A quantitative version of Pólya–Szegő inequality is proven for log-concave functions in the case of Steiner and Schwarz rearrangements.

1. Introduction

The Pólya–Szegő principle states that, given a non-negative function $u : \mathbb{R}^n \to \mathbb{R}$, the Dirichlet integral $\int_{\mathbb{R}^n} |\nabla u|^p$ decreases under suitable rearrangements, the two most common of which are the Schwarz spherical symmetrization about a point and Steiner symmetrization about a hyperplane. Their corresponding Pólya–Szegő inequalities are

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a powerful tool to approach a wide number of variational problems of geometric and functional nature.

Although the Pólya–Szegő inequality is known from long time, the issue of characterizing the extremals has been studied only more recently. In particular the first characterization of equality cases in the Pólya–Szegő inequality for spherical rearrangements has been provided by Brothers and Ziemer in [7] (see also [15] for an alternative proof). Instead, for Steiner symmetrization the characterization has been obtained in [13]. Both these results have been extended to the intermediate codimensions in [9].

When compared with these results, the natural issue of proving quantitative versions of the Pólya–Szegő inequality is a much more delicate task. The reason is that, even when the Dirichlet integral of a function u and of its symmetral coincide, u can be very different from its symmetral. In the case of Schwarz symmetrization this phenomenon may appear when the gradient of u is zero on sets of positive measure. Similarly, in the case of Steiner symmetrization the phenomenon could appear as soon as the derivative of u in the direction orthogonal to the symmetrization hyperplane is zero on sets of positive measure. Therefore any stability result for the Pólya–Szegő inequality must require a control of the measure of the set where the gradient or some of the derivatives are small (see [13,11] and the examples therein).

In this paper we deal with the stability of the Pólya–Szegő inequality for the symmetrization of functions having at least a mild form of concavity that is a natural geometric compromise to avoid the phenomena described above. At the same time the class of functions to which our stability results apply is large enough to include the solutions of the torsion problem and the first eigenfunction of the Laplacian operator with Dirichlet boundary conditions in smooth convex domains (see [5,21,20]).

To describe our main results let us recall the definition of Steiner symmetrization for a measurable function $u : \mathbb{R}^n \to [0, \infty)$ with compact support. For simplicity we write a point $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The Steiner symmetral u^s of u with respect to the hyperplane $\{x_n = 0\}$ is defined as

$$u^s(x) := \inf\bigl\{t > 0 \ :\ \mathcal{L}^1\bigl(\bigl\{s \ :\ u\bigl(x', s\bigr) > t\bigr\}\bigr) \leqslant 2|x_n|\bigr\}$$

for any $x \in \mathbb{R}^n$. If the function u belongs to $W^{1,p}(\mathbb{R}^n)$, then also u^s belongs to the same space and the Pólya–Szegő inequality states that

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geqslant \int_{\mathbb{R}^n} \bigl|\nabla u^s\bigr|^p \, dx.$$

Let us denote by

$$\Delta\bigl(u, u^s\bigr) := \int_{\mathbb{R}^n} |\nabla u|^p \, dx - \int_{\mathbb{R}^n} \bigl|\nabla u^s\bigr|^p \, dx$$