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The structure of almost-invariant half-spaces for some operators

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ABSTRACT

Given a Banach space X and an operator $T \in \mathcal{L}(X)$, an almost-invariant half-space (AIHS) is a closed subspace Y satisfying $TY \subseteq Y + F$ for some finite-dimensional subspace F and $\dim(Y) = \dim(X/Y) = \infty$. In this paper we study the connection between AIHS and noncompactness of some subsets of the unit ball, chains of T-invariant subspaces, and different parts of spectrum of T and T^*. We describe a simple template for such subspaces and show that if T is quasinilpotent or weakly compact then it admits an AIHS Y.

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1. Introduction

One of the recent approaches to the famous Invariant Subspace Problem has been the concept of almost-invariant half-spaces introduced in [4] in 2009. Although there have been negative and positive results on the Invariant Subspace Problem, the complete picture is far from being finished.

Bounded operators without invariant subspaces have been discovered for quite a while: Read [18] and Enflo [7] built the first examples while some of the latest could be seen in papers by Grivaux and Roginskaya in [8,9]. A simplified version of Read's construction

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can also be found in [19]. The first Banach space on which every bounded operator has an invariant subspace has been recently constructed in [5] by Argyros and Haydon as a solution to the scalar-plus-compact problem.

In this paper we mostly concentrate on an independent though related question if every bounded operator on a Banach space has an almost-invariant half-space (hereafter "AIHS"). This subject has been introduced and studied in the recent papers [4,16,15,17] for single operators as well as for collections of operators. For instance, in [15] the authors proved that every norm-closed algebra of operators on a Hilbert space admits an invariant halfspace whenever joint AIHS exists. In [16], Popov showed that the dimensions of defect spaces of a common AIHS for a norm-closed algebra of Banach space operators are uniformly bounded. Eventually, in [17], Popov and Tcaciuc proved that every bounded operator on a reflexive Banach space has an AIHS.

The core of this paper consists of Sections 2 and 3. After the introduction, in Section 2, we turn our attention to the fact that all specific constructions of AIHS in [16,4,14,17] were pretty much of two types. They were either spanned by a sequence of eigenvectors or else by a sequence $((\lambda_n - T)^{-1}e)_{n=1}^{\infty}$ for $(\lambda_n) \subset \rho(T)$. We explore this idea and connect almost-invariant half-spaces with noncompactness of some subset of the closed unit ball. Next, in Section 3, we concentrate on the point spectrum of the adjoint operator T^* and its connection to AIHS's. There we prove the main result of this paper that every quasinilpotent operator on an infinite-dimensional Banach space admits an AIHS of defect ≤ 1.

Presently, we state some standard notation and definitions and set some agreements for this paper. For the rest of our paper, we assume that all operators are linear and continuous. We agree that X will always be an infinite-dimensional complex Banach space, and denote the space of all continuous linear operators between two Banach spaces X and Y by $\mathcal{L}(X,Y)$ with agreement $\mathcal{L}(X,X) =: \mathcal{L}(X)$. $B_X := \{x \in X : \|x\| \leq 1\}$ is the closed unit ball in X, and $S_X := \{x \in X : \|x\| = 1\}$ the unit sphere. For a closed subspace Y of X, the **codimension** of Y in X is defined as the dimension of X/Y. A **halfspace**, then, is a closed subspace with both infinite dimension and codimension. A linear subspace Y of X is called ***T*-almost-invariant** (or, "almost-invariant under T") just in case there exists a finite-dimensional subspace F of X such that $TY \subseteq Y + F$. Let F be such that it has minimum dimension; then F is called the **error** of Y with respect to T, and the dimension of F is called the **defect**, written $d_{Y,T}$. We will use the acronyms **AIHS** and **IHS** for "almost-invariant halfspace" and "invariant halfspace," respectively. We will denote the linear span of a subset $S \subseteq X$ as $\operatorname{span} S$, and the closure of S and $\operatorname{span} S$ as $\overline{S}$ and $\overline{\operatorname{span}}\, S$, respectively. In place of $\overline{\operatorname{span}}\{x_n\}_{n=0}^{\infty}$ we may sometimes write $[x_n]_{n=0}^{\infty}$; similarly, for $x \in X$ we write $[x] := \operatorname{span}\{x\}$. We also define $\mathcal{O}(x) := \operatorname{span}\{T^i x\}_{i=0}^{\infty}$, i.e. the orbit space of x under T. We denote the image of T by $\operatorname{Im} T$ and the null space by $\mathcal{N}(T) := \{x : Tx = 0\}$. If Y is a linear subspace of X, then we denote by $T|_Y$ the restriction of T to Y. As usual, we denote the spectrum of T by $\sigma(T)$, the point spectrum of T by $\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(\lambda - T) \neq 0\}$, and the surjectivity spectrum by $\sigma_{\mathrm{su}}(T) := \{\lambda \in \mathbb{C} : (\lambda - T)X \neq X\}$. Of course, $\rho(T)$, $\rho_p(T)$, and $\rho_{\mathrm{su}}(T)$ denote their