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Korn type inequalities in Orlicz spaces

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We exhibit balance conditions between a Young function A and a Young function B for a Korn type inequality to hold between the L^B norm of the gradient of vector-valued functions and the L^A norm of its symmetric part. In particular, we extend a standard form of the Korn inequality in L^p, with $1 < p < \infty$, and an Orlicz version involving a Young function A satisfying both the Δ_2 and the ∇_2 condition.

1. Introduction

Given an open subset Ω of $\mathbb{R}^n$ and a vector-valued function $\mathbf{u} : \Omega \to \mathbb{R}^n$, the (distributional) symmetric gradient $\mathcal{E}\mathbf{u}$ of $\mathbf{u}$ is defined as the symmetric part of its gradient $\nabla\mathbf{u} \in \mathbb{R}^{n\times n}$. Here, $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices. In formulas,

$$\mathcal{E}u = \frac{1}{2}\big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\big),$$

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where $(\nabla\mathbf{u})^T$ stands for the transpose of $\nabla\mathbf{u}$. Sobolev type spaces $E^{1,p}(\Omega,\mathbb{R}^n)$, with $1 \le p \le \infty$, where the role of the full gradient in the standard Sobolev spaces $W^{1,p}(\Omega,\mathbb{R}^n)$ is instead played by the symmetric gradient, can be defined as

$$E^{1,p}\big(\Omega,\mathbb{R}^n\big) = \big\{\mathbf{u} \in L^p\big(\Omega,\mathbb{R}^n\big) : \text{the distribution } \mathcal{E}\mathbf{u} \text{ belongs to } L^p\big(\Omega,\mathbb{R}^{n\times n}\big)\big\}. \quad (1.1)$$

Similarly, the space $E^{1,p}_0(\Omega,\mathbb{R}^n)$ of those functions in $E^{1,p}(\Omega,\mathbb{R}^n)$ which vanish, in a suitable sense, on $\partial\Omega$, replaces the Sobolev space $W^{1,p}_0(\Omega,\mathbb{R}^n)$. The space of those functions $\mathbf{u}$ such that $\mathcal{E}\mathbf{u}$ is just a finite matrix-valued Radon measure on Ω, called functions of bounded deformation, has also been investigated in the literature [5,29,53].

A fundamental result in the theory of spaces built upon the symmetric gradient is the Korn inequality. Standard references for this inequality include [16,21,23,24,26,27,34,35,42,43]; more recent developments along various directions can be found in [2–4,6,8,13,18,30,37,44–47,55]. In its basic form, the Korn inequality asserts that, if Ω is bounded and $1 < p < \infty$, then there exists a constant C such that

$$\|\nabla\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le C\|\mathcal{E}\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \quad (1.2)$$

for every $\mathbf{u} \in E^{1,p}_0(\Omega,\mathbb{R}^n)$. A version of inequality (1.2) for functions which need not vanish on $\partial\Omega$ tells us that if Ω is a Lipschitz domain and $1 < p < \infty$, then there exists a constant C such that

$$\inf_{\mathbf{S}=-\mathbf{S}^T} \|\nabla\mathbf{u} - \mathbf{S}\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \le C\|\mathcal{E}\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^{n\times n})} \quad (1.3)$$

for every $\mathbf{u} \in E^{1,p}(\Omega,\mathbb{R}^n)$. Recall that a Lipschitz domain is a bounded connected open set in $\mathbb{R}^n$ which, in a neighborhood of each point of its boundary, agrees with the subgraph of a Lipschitz continuous function of $n-1$ variables in a suitable orthogonal coordinate system. Let us notice that inequality (1.3) can be equivalently stated with the infimum extended over all matrices $\mathbf{S} \in \mathbb{R}^{n\times n}$, instead of just those matrices $\mathbf{S}$ which are skew-symmetric – see Remark 3.7, Section 3.

Inequalities (1.2) and (1.3) roughly amount to asserting that gradients, whose symmetric part is small in a Lebesgue norm, are close, in the same norm, to a constant skew-symmetric matrix, and, in the case of functions vanishing on the boundary, such a matrix vanishes.

The spaces $E^{1,p}(\Omega,\mathbb{R}^n)$, and variants of them, are of crucial use in the analysis of mathematical models for certain physical phenomena described in terms of differentiable vector-valued functions $\mathbf{u}$, and depending just on their symmetric gradient $\mathcal{E}\mathbf{u}$. This is the case, for instance, in a classical mathematical theory of plasticity, where $\mathbf{u}(x)$ stands for the displacement from the unconstrained equilibrium position of a plastic body at the point x, and $\|\mathcal{E}\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^{n\times n})}$ (or, more generally, the total variation of $\mathcal{E}\mathbf{u}$ if the latter is just a finite Radon measure) accounts for the energy of the deformation [53].