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G-convergence, Dirichlet to Neumann maps and invisibility [☆]

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ABSTRACT

We establish optimal conditions under which the G-convergence of linear elliptic operators implies the convergence of the corresponding Dirichlet to Neumann maps. As an application we show that the approximate cloaking isotropic materials from [19] are independent of the source.

1. Introduction

We start with the definition of the Dirichlet to Neumann map (voltage to current) map. Given a bounded domain $\Omega \in \mathbb{R}^d$ and an elliptic matrix $\sigma \in L^\infty(\Omega, \mathcal{M}^{d\times d})$,

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for a given boundary data $\varphi \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem;

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi. \end{cases} \tag{1.1}$$

When the boundary is sufficiently smooth, the measurements on the boundary consist of the classical Dirichlet-to-Neumann map

$$\Lambda_\sigma(\varphi) = \langle \sigma \nabla u, \nu \rangle|_{\partial\Omega}, \tag{1.2}$$

where ν denotes the exterior unit normal to the boundary. In this way $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. It follows by integration by parts that Λ_σ can also be described in the weak form as

$$\langle \Lambda_\sigma(\varphi), \psi \rangle = \int_\Omega \langle \sigma \nabla u, \nabla \tilde{\psi} \rangle, \tag{1.3}$$

where $\psi \in H^{1/2}(\partial\Omega)$ and $\tilde{\psi} \in H^1(\Omega)$ is an extension of ψ into Ω. In case $\partial\Omega$ lacks of a proper normal, the weak formulation is still valid.

The Calderón inverse problem consists of the stable determination of σ from Λ_σ, see [32,21,29,7] for the uniqueness in the isotropic case, [1,2,9,10,13,15,12] for stability and [28,29] for the reconstruction. Much less is known in the anisotropic case except in dimension $d = 2$ [8]. Notice that when the Dirichlet to Neumann map is known for all energies, uniqueness and stability are studied also for the anisotropic case, see e.g. [23,6].

The results from [1,9,10,13,15,12] require some uniform control of the oscillations of σ (conditional stability). Unfortunately, wild oscillations of a sequence of conductivities σ_n create instability of the Calderón problem. This is well expressed in terms of the G-topology [14,22]. It is not hard to see that if σ_n G-converges to σ, the corresponding Dirichlet to Neumann maps converge weakly. Namely, for each $\varphi, \psi \in H^{1/2}(\partial\Omega)$,

$$\langle \Lambda_{\sigma_n}(\varphi), \psi \rangle \to \langle \Lambda_\sigma(\varphi), \psi \rangle. \tag{1.4}$$

Now, if σ_n G-converges to σ but does not convergence pointwise, we deduce that the convergence (1.4) does not imply any sort of L^p convergence. (Notice σ_n, σ could be chosen to be C^∞!)

However, the stability estimates are normally stated in terms of the operator norm and (1.4) by itself does not imply the convergence in the operator norm $\|\|_{\mathcal{L}(H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega))}$. In [4], it is proved that if, in addition to the G-convergence, we have that $\sigma_n = \sigma = I$ on $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$, with Ω being the unit disc and $\sigma = I$, then in fact the G-convergence implies the convergence in the operator norm. On the other hand, the stability at the boundary of the inverse problem implies that, in order to obtain operator norm convergence, some control on the behaviour of