Journal of Functional Analysis 269 (2015) 2305–2327

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Remarks on the Cauchy problem for the one-dimensional quadratic (fractional) heat equation

Luc Molinet [a,*], Slim Tayachi [b]

[a] *Laboratoire de Mathématiques et Physique Théorique, Universté Francois Rabelais Tours, CNRS UMR 7350, Fédération Denis Poisson, Parc Grandmont, 37200 Tours, France*

[b] *Department of Mathematics, Faculty of Science of Tunis, University Tunis El Manar, 2092 Tunis, Tunisia*

A R T I C L E   I N F O

Article history:
Received 27 May 2013
Accepted 1 August 2015
Available online 18 August 2015
Communicated by I. Rodnianski

MSC:
35K15
35K55
35K65
35B40

Keywords:
Nonlinear heat equation
Fractional heat equation
Ill-posedness
Well-posedness

A B S T R A C T

We prove that the Cauchy problem associated with the one dimensional quadratic (fractional) heat equation: $u_t = -(-\Delta)^\alpha u \mp u^2$, $t \in (0,T)$, $x \in \mathbb{R}$ or $\mathbb{T}$, with $0 < \alpha \leq 1$ is well-posed in H^s for $s \geq \max(-\alpha, 1/2 - 2\alpha)$ except in the case $\alpha = 1/2$ where it is shown to be well-posed for $s > -1/2$ and ill-posed for $s = -1/2$. As a by-product we improve the known well-posedness results for the heat equation ($\alpha = 1$) by reaching the end-point Sobolev index $s = -1$. Finally, in the case $1/2 < \alpha \leq 1$, we also prove optimal results in the Besov spaces $B_2^{s,q}$.

* Corresponding author.
E-mail addresses: Luc.Molinet@lmpt.univ-tours.fr (L. Molinet), slim.tayachi@fst.rnu.tn (S. Tayachi).

http://dx.doi.org/10.1016/j.jfa.2015.08.002

1. Introduction and main results

The Cauchy problem for the quadratic fractional heat equation reads

$$u_t + (-\Delta)^\alpha u = \mp u^2, \tag{1.1}$$

$$u(0, \cdot) = u_0, \tag{1.2}$$

where $u = u(t, x) \in \mathbb{R}$, $\alpha \in]0, 1]$, $t \in (0, T)$, $T > 0$, $x \in \mathbb{R}$ or $\mathbb{T}$ and $(-\Delta)^\alpha$ is the Fourier multiplier by $|\xi|^{2\alpha}$. In this paper, we consider actually the corresponding integral equation which is given by

$$u(t) = S_\alpha(t)u_0 \mp \int_0^t S_\alpha(t - \sigma)\big(u^2(\sigma)\big)d\sigma, \tag{1.3}$$

where $S_\alpha(t)$ is the linear fractional heat semi-group and are interested in local well-posedness and ill-posedness results in the Besov spaces $B_2^{s,q}(K)$ with $s \in \mathbb{R}$, $q \in [1, \infty[$ and $K = \mathbb{R}$ or $\mathbb{T}$.

Let us recall that the Cauchy problem associated with the nonlinear heat equation in $\mathbb{R}^n$

$$u_t - \Delta u = \mp u^k\,, \tag{1.4}$$

where k is a positive integer, has been studied in many papers (see for instance [2–6, 8,10–13,15–18] and references therein). It is well-known that this equation is invariant by the space–time dilation symmetry $u(t, x) \mapsto u_\lambda(t, x) = \lambda^{\frac{2}{k-1}} u(\lambda^2 t, \lambda x)$ and that the homogeneous Sobolev space $\dot{H}^{\frac{n}{2} - \frac{2}{k-1}}$ is invariant by the associated space dilation symmetry $\varphi(x) \mapsto \lambda^{\frac{2}{k-1}} \varphi(\lambda x)$. The Cauchy problem (1.4) is known to be well-posed in H^s for $s > s_c = \frac{n}{2} - \frac{2}{k-1}$ except in the case $(n, k) = (1, 2)$. Indeed, in this case the well-posedness is only known in H^s for $s > -1$ and in [8] it is proven that the flow-map cannot be of class C^2 below H^{-1}. Hence, this result is close to be optimal if one requires the smoothness of the flow-map. Recently, it was proven in [7] that the associated solution-map: $u_0 \mapsto u$ cannot be even continuous in H^s for $s < -1$. The first aim of this work is to push down the well-posedness result to the end point H^{-1}. The second step is to extend these type of results for the one-dimensional quadratic fractional heat equation (1.1). Indeed we will derive optimal results for the Cauchy problem (1.1) in the scale of the Besov spaces $B_2^{s,q}$ in the case $\frac{1}{2} < \alpha \leq 1$. In particular we will prove that the lowest reachable Sobolev index is $-\alpha$ that is strictly bigger than the critical Sobolev index for dilation symmetry that is $1/2 - 2\alpha$.

To reach the end-point index $H^{-\alpha}$ we do not follow the classical method for parabolic equations (cf. [3,10,18]) that does not seem to be applicable here. We rather rely on an approach that was first introduced by Tataru [14] in the context of wave maps. Note that we mainly follow [9] where this method has been adapted for dispersive–dissipative