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Ergodic and mixing properties of the Boussinesq equations with a degenerate random forcing

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A B S T R A C T

We establish the existence, uniqueness and attraction properties of an ergodic invariant measure for the Boussinesq equations in the presence of a degenerate stochastic forcing acting only in the temperature equation and only at the largest spatial scales. The central challenge is to establish time asymptotic smoothing properties of the Markovian dynamics corresponding to this system. Towards this aim we encounter a Lie bracket structure in the associated vector fields with a complicated dependence on solutions. This leads us to develop a novel Hörmander-type condition for infinite-dimensional systems. Demonstrating the sufficiency of this condition requires

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new techniques for the spectral analysis of the Malliavin covariance matrix.

Contents

1. Introduction

In this work we analyze the stochastically forced Boussinesq equations for the velocity field $\mathbf{u} = (u_1, u_2)$, (density-normalized) pressure p, and temperature θ of a viscous incompressible fluid. These equations take the form

$$d\mathbf{u} + (\mathbf{u} \cdot \nabla \mathbf{u})dt = (-\nabla p + \nu_1 \Delta \mathbf{u} + \mathbf{g}\theta)dt, \quad \nabla \cdot \mathbf{u} = 0, \tag{1.1}$$

$$d\theta + (\mathbf{u} \cdot \nabla \theta)dt = \nu_2 \Delta \theta dt + \sigma_\theta dW, \tag{1.2}$$

where the parameters $\nu_1, \nu_2 > 0$ are respectively the kinematic viscosity and thermal diffusivity of the fluid and $\mathbf{g} = (0, g)^T$ with $g \neq 0$ is the product of the gravitational constant and the thermal expansion coefficient. The spatial variable $x = (x_1, x_2)$ belongs to a two-dimensional torus $\mathbb{T}^2$. That is, we impose periodic boundary conditions in space. We consider a degenerate stochastic forcing $\sigma_\theta dW$, which acts only on a few Fourier modes and exclusively through the temperature equation.

We prove that there exists a unique statistically invariant state of the system (1.1)–(1.2). More precisely, we establish:

Theorem 1.1. *With white noise acting only on the two largest standard modes of the temperature equation* (1.2),

$$\sigma_\theta dW = \alpha_1 \cos x_1 dW^1 + \alpha_2 \sin x_1 dW^2 + \alpha_3 \cos x_2 dW^3 + \alpha_4 \sin x_2 dW^4,$$

the Markov semigroup corresponding to (1.1)–(1.2) *possesses a unique ergodic invariant measure. Moreover this measure is mixing, and it obeys a law of large numbers and a central limit theorem.*