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Saturation and elementary equivalence of C*-algebras

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A B S T R A C T

We study the saturation properties of several classes of C*-algebras. Saturation has been shown by Farah and Hart to unify the proofs of several properties of coronas of σ-unital C*-algebras; we extend their results by showing that some coronas of non-σ-unital C*-algebras are countably degree-1 saturated. We then relate saturation of the abelian C*-algebra $C(X)$, where X is 0-dimensional, to topological properties of X, particularly the saturation of $CL(X)$. We also characterize elementary equivalence of the algebras $C(X)$ in terms of $CL(X)$ when X is 0-dimensional, and show that elementary equivalence of the generalized Calkin algebras of densities $\aleph_\alpha$ and $\aleph_\beta$ implies elementary equivalence of the ordinals α and β.

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1. Introduction

In this paper we examine extent to which several classes of operator algebras are saturated in the sense of model theory. In fact, few operator algebras are saturated in the full model-theoretic sense, but in this setting there are useful weakenings of saturation that are enjoyed by a variety of algebras. The main results of this paper show that certain classes of C*-algebras do have some degree of saturation, and as a consequence, have a variety of properties previously considered in the operator algebra literature. For all the definitions involving continuous model theory for metric structures (and in particular of C*-algebras), we refer to [5] or [21]. Different degrees of saturation and relevant concepts will be defined in Section 2.

Among the weakest possible kinds of saturation an operator algebra may have, which nevertheless has interesting consequences, is being *countably degree*-1 *saturated*. This property was introduced by Farah and Hart in [20], where it was shown to imply a number of important consequences (see Theorem 2.8 below). It was also shown in [20] that countable degree-1 saturation is enjoyed by a number of familiar algebras, such as coronas of σ-unital C*-algebras and all non-trivial ultraproducts and ultrapowers of C*-algebras. Further examples were found by Voiculescu [37]. Countable degree-1 saturation can thus serve to unify proofs about these algebras. We extend the results of Farah and Hart by showing that a class of algebras which is broader than the class of σ-unital ones have countably degree-1 saturated coronas. The following theorem is Theorem 3.8 below; for the definitions of σ-unital C*-algebras and essential ideals, see Definition 3.1.

Theorem A. *Let M be a unital C*-algebra, and let $A \subseteq M$ be an essential ideal. Suppose that there is an increasing sequence of positive elements in A whose supremum is 1_M, and suppose that any increasing uniformly bounded sequence converges in M. Then M/A is countably degree-*1 *saturated.*

One interesting class of examples of a non-σ-unital algebra to which our result applies is the following. Let N be a II_1 factor, H a separable Hilbert space and $\mathcal{K}$ be the unique two-sided closed ideal of the von Neumann tensor product $N \mathbin{\overline{\otimes}} \mathcal{B}(H)$ (see [8] and [9]). Then $(N \mathbin{\overline{\otimes}} \mathcal{B}(H))/\mathcal{K}$ is countably degree-1 saturated. These results are the contents of Section 3.

In Section 4 we consider generalized Calkin algebras of uncountable weight, as well as $\mathcal{B}(H)$ where H has uncountable density. Considering their complete theories as metric structures, we obtain the following (Theorem 4.3 below):

Theorem B. *Let $\alpha \neq \beta$ be ordinals, H_α the Hilbert space of density $\aleph_\alpha$. Let $\mathcal{B}_\alpha = \mathcal{B}(H_\alpha)$ and $\mathcal{C}_\alpha = \mathcal{B}_\alpha/\mathcal{K}$ the Calkin algebra of density $\aleph_\alpha$. Then the projections of the algebras $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ as posets with respect to the Murray–von Neumann order are elementary equivalent if and only if $\alpha = \beta \mod \omega^\omega$, where ω^ω is computed by ordinal exponentiation,*