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Spectral and asymptotic properties of Grover walks on crystal lattices

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ABSTRACT

We propose a twisted Szegedy walk for estimating the limit behavior of a discrete-time quantum walk on a crystal lattice, an infinite abelian covering graph, whose notion was introduced by [14]. First, we show that the spectrum of the twisted Szegedy walk on the quotient graph can be expressed by mapping the spectrum of a twisted random walk onto the unit circle. Secondly, we show that the spatial Fourier transform of the twisted Szegedy walk on a finite graph with appropriate parameters becomes the Grover walk on its infinite abelian covering graph. Finally, as an application, we show that if the Betti number of the quotient graph is strictly greater than one, then localization is ensured with some appropriated initial state. We also compute the limit density function for the Grover walk on $\mathbb{Z}^d$ with flip flop shift, which implies the coexistence of linear spreading and localization. We partially obtain the abstractive shape of the limit density function: the support is within the d-dimensional

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sphere of radius $1/\sqrt{d}$, and 2^d singular points reside on the sphere's surface.

1. Introduction

Quantum walks have been intensively studied from various perspectives. A primitive form of a discrete-time quantum walk on $\mathbb{Z}$ can be found in the so called Feynman's checker board [3]. An actual discrete-time quantum walk itself was introduced in a study of quantum probabilistic theory [4]. The Grover walk on general graphs was proposed in [20]. This quantum walk has been intensively-investigated in quantum information theory and spectral graph theory [1,7], and is known to accomplish a quantum speed up search in certain cases. A review on applications of quantum walks to the quantum search algorithms is presented in [1]. To enable more abstract interpretation of these quantum search algorithms, the Grover walk was generalized to the Szegedy walk in 2004 [18]. One advance of the Szegedy walk is that the performance of the quantum search algorithm based on the Szegedy walk is usually evaluated by hitting time of the underlying random walk.

In this paper, we introduce a "twisted" Szegedy walk on $\ell^2(D)$ for a given graph $G = (V, D)$, where V and D are the sets of vertices and arcs, respectively. In parallel, we also present a twisted (random) walk on $\ell^2(V)$ underlying the quantum walk on $\ell^2(D)$ for providing a spectral mapping theory. Study on this twisted random walks has been well developed; the effects of some spatial structures of the crystal lattice on the return probability and the central limit theorems of the random walks were clarified [13,14], for example. To elucidate the relationship between the spectra of the twisted random walk and the twisted Szegedy walk, we introduce new boundary operators $d_A, d_B : \ell^2(D) \to \ell^2(V)$. Then we show that $\mathcal{L} \equiv d_A^*(\ell^2(V)) + d_B^*(\ell^2(V)) \subset \ell^2(D)$ is invariant under the action of the twisted Szegedy walk. Here $d_A^*, d_B^* : \ell^2(V) \to \ell^2(D)$ are the adjoint operators of d_A and d_B, respectively. Careful observation reveals that the eigenvalues of the twisted random walk on $\ell^2(V)$ describe the "real parts" of the eigenvalues of the quantum walk (Proposition 1). Thus we call the eigenspace $\mathcal{L}$ the inherited part from the twisted random walk. The remainder of the eigenspace; $\mathcal{L}^\perp$, is expressed by the intersection of the kernels of the boundary operators d_A and d_B. For a typical boundary operator of graphs $\partial : C_1 \to C_0$, $\ker(\partial)$ is generated by all closed paths of G. Here $C_1 = \sum_{e\in D(G)} \mathbb{Z}\delta_e$ and $C_0 = \sum_{v\in V(G)} \mathbb{Z}\delta_v$. We show that the orthogonal complement space $\mathcal{L}^\perp = \ker(d_A)\cap\ker(d_B)$ is also characterized by all closed paths of G (Theorem 1). This fact implies that there is a homological abstraction within the Grover walk, and this abstraction is crucial to provide a typical stochastic behavior called localization (see Theorem 2).

There are many types of mapping theorems. For example, Higuchi and Shirai provided how the spectra of the Laplacian changes under graph-operations in [6]. They mapped the spectrum of the Laplacian of the original graph G to those of the line graph LG, the