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To the theory of viscosity solutions for uniformly elliptic Isaacs equations

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We show how a theorem about the solvability in $C^{1,1}$ of special Isaacs equations can be used to obtain existence and uniqueness of viscosity solutions of general uniformly nondegenerate Isaacs equations. We apply it also to establish the $C^{1+\chi}$ regularity of viscosity solutions and show that finite-difference approximations have an algebraic rate of convergence. The main coefficients of the Isaacs equations are supposed to be in C^{γ} with γ slightly less than 1/2.

1. Introduction

The goal of this article is to present a *purely PDE* exposition of some major results in the theory of viscosity solutions for uniformly nondegenerate Isaacs equations.

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Let $\mathbb{R}^d = \{x = (x^1, ..., x^d)\}$ be a d-dimensional Euclidean space. Assume that we are given separable metric spaces A and B, and let, for $(\alpha, \beta, x) \in A \times B \times \mathbb{R}^d$, the following functions be given:

(i) $d \times d$ matrix-valued $a^{\alpha\beta}(x)$,
(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(x)$, and
(iii) real-valued functions $c^{\alpha\beta}(x) \geq 0$, $f^{\alpha\beta}(x)$, and $g(x)$.

Let $\mathbb{S}$ be the set of symmetric $d \times d$ matrices, and for $(u_{ij}) \in \mathbb{S}$, $(u_i) \in \mathbb{R}^d$, and $u \in \mathbb{R}$ introduce

$$F(u_{ij}, u_i, u, x) = \sup_{\alpha \in A} \inf_{\beta \in B} \big[a_{ij}^{\alpha\beta}(x) u_{ij} + b_i^{\alpha\beta}(x) u_i - c^{\alpha\beta}(x) u + f^{\alpha\beta}(x)\big],$$

where and everywhere below the summation convention is enforced and the summations are done inside the brackets.

For a sufficiently smooth function $u = u(x)$ also introduce

$$L^{\alpha\beta} u(x) = a_{ij}^{\alpha\beta}(x) D_{ij} u(x) + b_i^{\alpha\beta}(x) D_i u(x) - c^{\alpha\beta}(x) u(x),$$

where, naturally, $D_i = \partial/\partial x^i$, $D_{ij} = D_i D_j$. Denote

$$F[u](x) = F\big(D_{ij} u(x), D_i u(x), u(x), x\big) = \sup_{\alpha \in A} \inf_{\beta \in B} \big[L^{\alpha\beta} u(x) + f^{\alpha\beta}(x)\big]. \tag{1.1}$$

Also fix a sufficiently regular domain $G \subset \mathbb{R}^d$. Under appropriate assumptions which we list in Section 2 and which include the boundedness and continuity with respect to x of the data and uniform nondegeneracy of $a^{\alpha\beta}(x)$ the Isaacs equation

$$F[u] = 0 \tag{1.2}$$

in G with boundary condition $u = g$ on ∂G has a viscosity solution $w \in C(\bar{G})$. Recall (see [4]) that this means that for any smooth $\phi(x)$ and any point $x_0 \in G$ at which $\phi - w$ attains

(i) a local maximum which is zero we have $F[\phi](x_0) \leq 0$,
(ii) a local minimum which is zero we have $F[\phi](x_0) \geq 0$.

We are going to discuss the existence, uniqueness, regularity properties of w, and the rate of convergence of finite-difference approximations to w and, therefore, we give a brief account of basic facts known for the Isaacs equations. We only discuss these equations although in the references below more general equations are considered and more details can be found. For brevity, when we mention that, say, a is uniformly continuous in x, we mean that a is continuous in x uniformity with respect to α, β, x. The Lipschitz