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Universal properties of group actions on locally compact spaces

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A B S T R A C T

We study universal properties of locally compact G-spaces for countable infinite groups G. In particular we consider open invariant subsets of the G-space βG, and their minimal closed invariant subspaces. These are locally compact free G-spaces, and the latter are also minimal. We examine the properties of these G-spaces with emphasis on their universal properties. As an example of our results, we use combinatorial methods to show that each countable infinite group admits a free minimal action on the locally compact non-compact Cantor set.

1. Introduction

Ellis proved in [2] that every group G admits a free minimal action on a compact Hausdorff space, and he proved that universal minimal G-spaces exist and are unique. (See also [3].) A (minimal) compact G-space is universal if any other (minimal) compact G-space is the image of the universal space by a continuous G-map. Ellis proved, more specifically, that each minimal closed invariant subset of the G-space βG is a free minimal

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G-space which is universal. The number of minimal closed invariant subsets of βG is very large, see [5], but they are all isomorphic by Ellis' uniqueness theorem.

Hjorth and Molberg, [6], established the existence of a free minimal action of any countable infinite group on the Cantor set (this is obtained from Ellis' results through a standard reduction argument). They also obtained a free action of an arbitrary countable group on the Cantor set admitting an invariant Borel probability measure.

The goal of this paper is to extend Ellis' results to the locally compact, non-compact setting, and in particular to study properties of the locally compact G-spaces that arise as *open* invariant subsets of βG, and their minimal *closed* invariant subsets (when they exist). We are particularly interested in those open invariant subsets of βG that give rise to co-compact G-spaces (they always contain minimal closed invariant subsets). These turn out to be of the form $X_A = \bigcup_{g \in G} K_{gA}$ for some subset A of G, where K_{gA} denotes the closure of the set gA in βG, and they are thus "indexed" by the set A. If A and B are subsets of G, then $X_A = X_B$ if and only if A is "B-bounded" and B is "A-bounded", or, equivalently, if the Hausdorff distance between A and B with respect to (any) proper right-invariant metric on G is finite.

The minimal closed invariant subspaces of a co-compact open invariant subspace of βG provide examples of locally compact free minimal G-spaces. We prove universality and uniqueness results for these spaces (explained in more detail below).

Kellerhals, Monod and the second named author studied actions of *supramenable* groups on locally compact spaces in [7]. By definition, a group is supramenable if it contains no (non-empty) paradoxical subset. It was shown in [7] that a group is supramenable if and only if whenever it acts *co-compactly* on a locally compact Hausdorff space, then there is a non-zero invariant Radon measure. As a step towards proving this result it was shown that there is a non-zero invariant Radon measure on the G-space X_A if and only if A is non-paradoxical. This is an example where a property of the G-space X_A is reflected in a property of the set A. We shall exploit such connections further in this paper. The condition that the action be co-compact in the characterization of supramenable groups from [7] cannot be removed as shown in Section 4.

It was further shown in [7] that if A is paradoxical, then any minimal closed invariant subset of X_A is a *purely infinite*[1] free minimal G-space. This was used to prove that any countable non-supramenable group admits a free minimal purely infinite action on the locally compact non-compact Cantor set. It was left open in [7] if *all* countable infinite groups admit a free minimal action on the locally compact non-compact Cantor set. In Section 8 we answer this question affirmatively for all countable infinite groups.

We show that the G-spaces X_A are universal with respect to the class of locally compact G-spaces that have the same *type* as X_A. The (base point dependent) type is defined for each pair (X, x_0), where X is a locally compact (co-compact) G-space such that $G.x_0$ is dense in X, and it is defined to be the collection of sets $B \subseteq G$ for which $B.x_0$

[1] An action of a group on a totally disconnected space is purely infinite if all compact-open subsets are paradoxical relatively to the compact-open subsets of the space.