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Are the incompressible 3d Navier–Stokes equations locally ill-posed in the natural energy space?

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A B S T R A C T

An important open problem in the theory of the Navier–Stokes equations is the uniqueness of the Leray–Hopf weak solutions with L^2 initial data. In this paper we give sufficient conditions for non-uniqueness in terms of spectral properties of a natural linear operator associated to scale-invariant solutions recently constructed in [8]. If the spectral conditions are satisfied, non-uniqueness and ill-posedness can appear for quite benign compactly supported data, just at the borderline of applicability of the classical perturbation theory. The verification of the spectral conditions seems to be approachable by numerical simulations which involve only smooth functions.

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1. Introduction

We consider the classical Cauchy problem for the 3d Navier–Stokes equations (NSE) in $R^3 \times (0, \infty)$,

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ u(\cdot, 0) &= u_0. \end{aligned} \tag{1.1}$$

Let us recall the scaling symmetry

$$\begin{aligned} u(x,t) &\to u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t), \\ u_0(x) &\to u_{0\lambda}(x) := \lambda u_0(\lambda x), \\ p(x,t) &\to p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t), \end{aligned}$$

defined for $\lambda > 0$. If u, p, u_0 satisfy the equations, so do $u_\lambda, p_\lambda, u_{0\lambda}$. A solution u is *scale invariant* if $u \equiv u_\lambda$ for all $\lambda > 0$. The initial condition u_0 is scale invariant if $u_0 \equiv u_{0\lambda}$ for all $\lambda > 0$. In [8], it was proved that for each scale invariant initial data $u_0 \in C^\alpha(R^3 \backslash \{0\})$ there exists at least one scale invariant solution $u \in C^\infty(R^3 \times (0, \infty)) \cap C^\alpha(R^3 \times [0, \infty) \backslash \{(0,0)\})$. See also [18] for a generalization to discretely self-similar solutions.

It has been conjectured in [7,8] that for many large scale-invariant initial data the scale invariant solutions are not unique, and possible implications for the non-uniqueness of the Leray–Hopf weak solutions were suggested. In this paper we investigate these topics further.

Let us consider a scale-invariant initial condition u_0 which is smooth away from the origin. For $\sigma \geq 0$, at first taken sufficiently small so that we have uniqueness, let $u_\sigma(x,t) = \frac{1}{\sqrt{t}} U_\sigma \left(\frac{x}{\sqrt{t}} \right)$ be the unique scale invariant solution to NSE with the initial data σu_0. Here and below we work with the class of Leray solutions [6,11].[1] The field U_σ satisfies

$$\Delta U_\sigma + \frac{x}{2} \cdot \nabla U_\sigma + \frac{1}{2} U_\sigma - U_\sigma \cdot \nabla U_\sigma + \nabla P = 0 \tag{1.2}$$

in R^3, with $|U_\sigma(x) - \sigma u_0(x)| = o(\frac{1}{|x|})$ as $x \to \infty$. By the results in [8] we know $|U_\sigma(x) - u_0(x)| = O(\frac{1}{|x|^3})$ as $|x| \to \infty$. We consider Navier–Stokes solutions $u(x,t)$ which are close to $u_\sigma(x,t)$ and write them as

[1] This class in particular specifies in which sense the initial condition is attained: $||u(t) - u_0||_{L^2(B)} \to 0$ as $t \to 0_+$ for each bounded ball $B \subset R^3$. In our case the initial condition is in fact attained in a stronger sense: $u(t)$ approaches u_0 locally uniformly in $R^3 \setminus \{0\}$ and in $L^p(B)$ with $p < 3$ for any bounded ball $B \subset R^3$.

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