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Moment measures

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ABSTRACT

With any convex function ψ on a finite-dimensional linear space X such that ψ goes to $+\infty$ at infinity, we associate a Borel measure μ on X^*. The measure μ is obtained by pushing forward the measure $e^{-\psi(x)}\,dx$ under the differential of ψ. We propose a class of convex functions – the essentially-continuous, convex functions – for which the above correspondence is in fact a bijection onto the class of finite Borel measures whose barycenter is at the origin and whose support spans X^*. The construction is related to toric Kähler–Einstein metrics in complex geometry, to Prékopa's inequality, and to the Minkowski problem in convex geometry.

1. Introduction

The aim of the present work is to extend the results on moment measures obtained by Berman and Berndtsson [3] in their work on Kähler–Einstein metrics in toric manifolds, which builds upon earlier works by Wang and Zhu [29], by Donaldson [10] and by E. Legendre [21]. Simultaneously, our analysis of moment measures should be viewed as a functional version of the classical Minkowski problem (see, e.g., Schneider [27, Section 7.1]) or the logarithmic Minkowski problem of Böröczky, Lutwak, Yang and

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Zhang [6]. Yet a third point of view is that we discuss a certain kind of Monge–Ampère equation, and establish existence and uniqueness of generalized solutions.

Suppose that $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function, i.e., for any $0 < \lambda < 1$ and $x, y \in \mathbb{R}^n$,

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y)$$

whenever $\psi(x) < +\infty$ and $\psi(y) < +\infty$. In this note, we treat $+\infty$ as a legitimate value of convex functions, and we use relations such as $\exp(-\infty) = 0$ whenever they make sense. The function ψ is locally-Lipschitz and hence differentiable almost everywhere in the interior of the set

$$\{\psi < +\infty\} = \{x \in \mathbb{R}^n\, ;\, \psi(x) < +\infty\}.$$

In Kähler geometry, the map

$$x \longrightarrow \nabla\psi(x),$$

defined almost-everywhere in $\{\psi < +\infty\}$, is closely related to the *moment map* of a toric Kähler manifold, see, e.g. Abreu [1] or Gromov [17]. When the function ψ is finite and smooth, the set

$$\nabla\psi(\mathbb{R}^n) = \{\nabla\psi(x); x \in \mathbb{R}^n\} \tag{1}$$

is necessarily convex. In certain cases of interest the convex set (1) is in fact a polytope, which is referred to as the *moment polytope*; a central role is played by a family of polytopes known as *Delzant polytopes* which carry a particular geometric structure.

In this article, we consider convex functions $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that satisfy the integrability condition $0 < \int \exp(-\psi) < \infty$. This condition, for a convex function ψ, is equivalent to the following two requirements:

(i) The convex set $\{\psi < +\infty\}$ is not contained in a hyperplane; and
(ii) $\lim_{|x|\to\infty} \psi(x) = +\infty$.

We associate with such ψ the finite (log-concave) measure ν_ψ on $\mathbb{R}^n$ whose density is $\exp(-\psi)$.

Definition 1. Given a convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $0 < \int \exp(-\psi) < \infty$, we define its moment measure μ to be the Borel measure on $\mathbb{R}^n$ which is the push-forward of ν_ψ under $\nabla\psi$. This means that

$$\int_{\mathbb{R}^n} b(y)\, d\mu(y) = \int_{\mathbb{R}^n} b(\nabla\psi(x))\, e^{-\psi(x)}\, dx \tag{2}$$

for every Borel function b such that $b \in L^1(\mu)$ or b is non-negative.