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The spectral density of a product of spectral projections

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A B S T R A C T

We consider the product of spectral projections

$$\Pi_\varepsilon(\lambda) = \mathbb{1}_{(-\infty,\lambda-\varepsilon)}(H_0)\mathbb{1}_{(\lambda+\varepsilon,\infty)}(H)\mathbb{1}_{(-\infty,\lambda-\varepsilon)}(H_0)$$

where H_0 and H are the free and the perturbed Schrödinger operators with a short range potential, $\lambda > 0$ is fixed and $\varepsilon \to 0$. We compute the leading term of the asymptotics of $\operatorname{Tr} f(\Pi_\varepsilon(\lambda))$ as $\varepsilon \to 0$ for continuous functions f vanishing sufficiently fast near zero. Our construction elucidates calculations that appeared earlier in the theory of "Anderson's orthogonality catastrophe" and emphasizes the role of Hankel operators in this phenomenon.

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1. Introduction

1.1. Motivation from mathematical physics

This paper is partly motivated by a phenomenon called "Anderson's orthogonality catastrophe", which has been intensively discussed in the physics literature and has recently attracted attention from a mathematical perspective; see [2,6,3] and the literature cited therein. Let H_0 and H be the free and the perturbed Schrödinger operators

$$H_0 = -\Delta\,, \qquad H = -\Delta + V \qquad \text{in } L^2(\mathbb{R}^d)\,, \quad d \geq 1\,,$$

where, for the sake of simplicity, the real-valued potential V is assumed to be bounded and compactly supported. For $\lambda > 0$, consider the product of spectral projections

$$\Pi(\lambda) = \mathbb{1}_{(-\infty,\lambda)}(H_0)\mathbb{1}_{(\lambda,\infty)}(H)\mathbb{1}_{(-\infty,\lambda)}(H_0) \quad \text{in } L^2(\mathbb{R}^d)\,. \tag{1.1}$$

One is interested in regularised versions of $\Pi(\lambda)$, obtained by replacing the step functions $\mathbb{1}_{(-\infty,\lambda)}$, $\mathbb{1}_{(\lambda,\infty)}$ by functions with disjoint supports. More precisely, we consider two types of regularisations of $\Pi(\lambda)$,

$$\Pi_\varepsilon^{(1)}(\lambda) = \mathbb{1}_{(-\infty,\lambda-\varepsilon)}(H_0)\mathbb{1}_{(\lambda+\varepsilon,\infty)}(H)\mathbb{1}_{(-\infty,\lambda-\varepsilon)}(H_0) \tag{1.2}$$

and

$$\Pi_\varepsilon^{(2)}(\lambda) = \psi_\varepsilon^-(H_0 - \lambda)\psi_\varepsilon^+(H-\lambda)\psi_\varepsilon^-(H_0-\lambda)\,, \tag{1.3}$$

where $\psi_\varepsilon^\pm$ are continuous functions on $\mathbb{R}$ that satisfy $0 \leq \psi_\varepsilon^\pm \leq 1$ and

$$\psi_\varepsilon^+(x) = \begin{cases} 0 & \text{if } x \leq \varepsilon\,, \\ 1 & \text{if } x \geq 2\varepsilon\,, \end{cases} \qquad \psi_\varepsilon^-(x) = \begin{cases} 1 & \text{if } x \leq -2\varepsilon\,, \\ 0 & \text{if } x \geq -\varepsilon\,. \end{cases} \tag{1.4}$$

It is not difficult to see that the operators $\Pi_\varepsilon^{(j)}(\lambda)$, $j = 1, 2$, are trace class. On the other hand, $\Pi(\lambda)$ is typically not trace class and not even compact. We discuss the asymptotics of traces

$$\operatorname{Tr} f(\Pi_\varepsilon^{(j)}(\lambda))\,, \qquad \varepsilon \to 0\,, \quad j = 1, 2\,, \tag{1.5}$$

where $f = f(t)$ is a continuous function which vanishes sufficiently fast as $t \to 0$.

It turns out that the asymptotics of the traces (1.5) is given in terms of the scattering matrix $S(\lambda)$ for the pair H_0, H at energy λ. Let $\{e^{i\theta_\ell(\lambda)}\}_{\ell=1}^L$ be the eigenvalues of $S(\lambda)$, enumerated with multiplicities taken into account. The scattering matrix is an operator in $L^2(\mathbb{S}^{d-1})$ for $d \geq 2$ and is a 2×2 matrix for $d = 1$; thus, $L = \infty$ for $d \geq 2$ and $L = 2$ for $d = 1$. Denote

$$a_\ell(\lambda) = \frac{1}{2}\left|e^{i\theta_\ell(\lambda)} - 1\right| = \left|\sin\frac{\theta_\ell(\lambda)}{2}\right| \in [0,1]\,, \qquad \ell = 1, \dots, L\,. \tag{1.6}$$