Journal of Functional Analysis 267 (2014) 3500–3506

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Note

Bounds for eigenvalues of Schatten–von Neumann operators via self-commutators

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A R T I C L E   I N F O

Article history:
Received 27 February 2014
Accepted 23 June 2014
Available online 14 August 2014
Communicated by A. Borodin

MSC:
47B10
47B06

Keywords:
Schatten–von Neumann operators
Inequality for eigenvalues

A B S T R A C T

Let H be a separable Hilbert space, A be a Schatten–von Neumann operator in H with the finite norm $N_{2p}(A) = [Trace(AA^*)^p]^{1/2p}$ for an integer $p \geq 1$ and $N_1(T) = Trace(TT^*)^{1/2}$ be the trace norm of a trace operator T. It is proved that

$$\sum_{k=1}^{\infty} \left|\lambda_k(A)\right|^{2p} \leq \left[N_{2p}^{4p}(A) - \frac{1}{4} N_1^2\left(\left[A, A^*\right]_p\right)\right]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, 2, ...)$ are the eigenvalues of A, $[A, A^*]_p = A^p(A^*)^p - (A^*)^p A^p$; A^* is the adjoint to A. This results refines the classical inequality $\sum_{k=1}^{\infty} |\lambda_k(A)|^{2p} \leq N_{2p}^{2p}(A)$. Lower bounds for $N_1([A, A^*]_p)$ are also suggested. In addition, if A is a Hilbert–Schmidt operator, we improve the well-known inequality

$$\sum_{k=1}^{\infty} \left|Im\, \lambda_k(A)\right|^2 \leq N_2^2(A_I),$$

where $A_I = (A - A^*)/2i$.

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http://dx.doi.org/10.1016/j.jfa.2014.06.019

1. Introduction and statements of the main results

Let H be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and the unit operator I.

For a linear operator A in H, A^* is the adjoint operator, $\sigma(A)$ is the spectrum, $\lambda_k(A)$ $(k = 1, 2, ...)$ are the eigenvalues of A with their multiplicities enumerated in the non-increasing order of their absolute values: $|\lambda_k(A)| \geq |\lambda_{k+1}(A)|$. By SN_p $(p = 1, 2, ...)$ we denote the Schatten–von Neumann ideal of compact operators A with the finite norm

$$N_p(A) := \left[Trace\left(AA^*\right)^{p/2}\right]^{1/p},$$

cf. [1,5,8]. In particular, $N_1(A)$ is the trace norm of a trace operator A and $N_2(A)$ is the Hilbert–Schmidt norm of $A \in SN_2$. In addition, $[A, A^*] = AA^* - A^*A$ is the self-commutator and

$$\left[A, A^*\right]_p = \left[A^p, \left(A^*\right)^p\right] = A^p\left(A^*\right)^p - \left(A^*\right)^p A^p.$$

So $[A, A^*] = [A, A^*]_1$.

The aim of present paper is to prove the following result.

Theorem 1.1. *For any $A \in SN_{2p}$ $(p = 1, 2, ...)$, one has*

$$\sum_{k=1}^{\infty}\left|\lambda_k(A)\right|^{2p} \leq \left[N_{2p}^{4p}(A) - \frac{1}{4}N_1^2\left(\left[A, A^*\right]_p\right)\right]^{1/2}. \tag{1.1}$$

All the proofs are presented in the next section. Theorem 1.1 refines the classical inequality

$$\sum_{k=1}^{\infty}\left|\lambda_k(A)\right|^{2p} \leq N_{2p}^{2p}(A), \tag{1.2}$$

cf. [5, Corollary II.3.1].

For any compact selfadjoint operator T whose entries in an orthogonal normal basis are t_{jk} $(j, k = 1, 2, ...)$ we have

$$\sum_{k=1}^{m}\left|\lambda_k(T)\right| \geq \sum_{k=1}^{m}|t_{jj}| \quad (m = 1, 2, ...), \tag{1.3}$$

cf. [5, Section II.4.3]. So, if $c_{jk}^{(p)}$ $(j, k = 1, 2, ...)$ are the entries of $[A, A^*]_p$ in an orthogonal normal basis, then

$$N_1\left(\left[A, A^*\right]_p\right) = \sum_{k=1}^{\infty}\left|\lambda_k\left(\left[A, A^*\right]_p\right)\right| \geq \sum_{k=1}^{\infty}\left|c_{jj}^{(p)}\right|.$$