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On mathematical foundation of the Brownian motor theory

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ABSTRACT

The paper contains mathematical justification of basic facts concerning the Brownian motor theory. The homogenization theorems are proved for the Brownian motion in periodic tubes with a constant drift. The study is based on an application of the Bloch decomposition. The effective drift and effective diffusivity are expressed in terms of the principal eigenvalue of the Bloch spectral problem on the cell of periodicity as well as in terms of the harmonic coordinate and the density of the invariant measure. We apply the formulas for the effective parameters to study the motion in periodic tubes with nearly separated dead zones.

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1. Introduction

The paper is devoted to mathematical theory of Brownian (molecular) motors. The concept of a Brownian motor has fundamental applications in the study of transport processes in living cells and (in a slightly different form) in the porous media theory. There are thousands of publications in the area of Brownian motors in the applied literature. For example, the review by P. Reimann [16] contains 729 references. Most of these publications are in physics or biology journals and are not mathematically rigorous (although many of them are based on the mathematical model described below, see, for example [4,5,10,20]). Some of them are based on numerical computations.

Consider a set of particles with an electrical charge performing Brownian motion in a tube Ω with periodic (or stationary random) cross section (see Fig. 1).

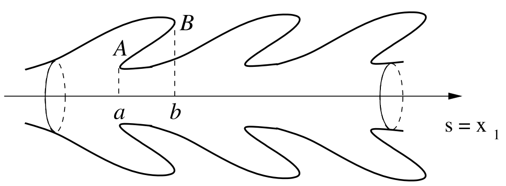

Fig. 1. A periodic tube Ω with “fingers”. One can expect that $V_{\text{eff}} < |V_{\text{eff}}^{-}|$.

We will assume that the axis of Ω is directed along the x_1-axis. Let’s apply a constant external electric field E along the axis of Ω. Then the motion of the particles consists of the diffusion and the Stocks drift, and the corresponding generator has the form

$$Lu = \Delta u + V\frac{\partial u}{\partial x_1}, \quad V = V(E),$$

complemented by the Neumann boundary condition on $\partial\Omega$ (we assume the normal reflection at the boundary).

One can expect that the displacement of the particles on the large time scale may be approximated (due to homogenization) by a one-dimensional diffusion process $\bar{x}_1(t)$ along the x_1-axis with an effective drift V_{eff} and an effective diffusivity σ^2, which depend on V and the geometry of the tube Ω. If we reverse the direction of the external field from E to $-E$, then the corresponding effective drift V_{eff}^{-} and the corresponding effective diffusivity $(\sigma^{-})^2$ will be, generally, different from V_{eff} and σ^2 when Ω is not symmetric with respect to the reflection $x_1 \to -x_1$. For example, one can expect that $V_{\text{eff}} < |V_{\text{eff}}^{-}|$ for Ω shown in Fig. 1. The difference between the effective parameters can be significant. Then by changing the direction of the exterior electric field E periodically in time, one could construct a constant drift in, say, the negative direction of x_1 and create a device (motor) producing energy from Brownian motion.

The idea of molecular motors goes back to M. Smoluchowski, R. Feynman, L. Brillouin. Starting from the 1990-s, it became a hot topic in chemical physics, molecular biology, and thermodynamics. The following natural problem must be solved by mathematicians: