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Uniqueness of signature for simple curves

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ABSTRACT

We propose a topological approach to the problem of determining a curve from its iterated integrals. In particular, we prove that a family of terms in the signature series of a two dimensional closed curve with finite p variation, $1 \leq p < 2$, are in fact moments of its winding number. This relation allows us to prove that the signature series of a class of simple non-smooth curves uniquely determine the curves. This implies that outside a Chordal SLE_κ null set, where $0 < \kappa \leq 4$, the signature series of curves uniquely determine the curves. Our calculations also enable us to express the Fourier transform of the n-point functions of SLE curves in terms of the expected signature of SLE curves. Although the techniques used in this article are deterministic, the results provide a platform for studying SLE curves through the signatures of their sample paths.

1. Introduction

The signature of a path is a formal series of its iterated integrals. In [6], K.T. Chen observed that the map that sends a path to its signature forms a homomorphism from

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the concatenation algebra to the tensor algebra and used it to study the cohomology of loop spaces. Recent interest in the study of signature has been sparked by its role in the rough path theory. In particular, it was shown by Hambly and Lyons in [11] that for ODEs driven by paths with bounded total variations, the signature is a fundamental representation of the effect of the driving signal on the solution.

This article has two purposes:

1. To determine the winding number of a curve from its signature.
2. To prove, using a relation obtained from answering 1., that the signature of sufficiently regular planar simple curves uniquely determines the curves.

The first question was originally considered as far back as 1936, in a paper by Rado [20], who observed that the second term of the signature series of a smooth path is equal to the integral of its winding number around (x, y), considered as a function of (x, y). In [29], Yam considered the same problem as ours, but used a different approach. He started with the formula

$$\text{Winding number around } z = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} \mathrm{d}w,$$

and smoothened the kernel $w \to \frac{1}{w-z}$ around the singularity at $w = z$. He then expanded $\frac{1}{w-z}$ into a power series of w and used the fact that the line integrals along γ of polynomials in w can be expressed in terms of the signature of γ.

Here we took a different approach and obtained a formula for the Fourier transform of the winding number, which appears to be simpler than the formula for the winding number itself. A classical result about iterated integrals, first proved by Chen [7], states that the logarithm of the signature of any path is a Lie series. The first result of this article states that the coefficients of some Lyndon basis elements in the log signature series are in fact moments of the winding number. In what follows, we will use some basic notions in free Lie algebra, which we shall recall in Section 3. Throughout this article, we will use π_N to denote the projection of $T((\mathbb{R}^d))$ to $T^N(\mathbb{R}^d)$ (see Section 2.1) and $S(\gamma)_{0,1}$ to denote the signature of γ.

Theorem 1. *Let $1 \le p < 2$. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous closed curve with finite p variation. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the standard basis of $\mathbb{R}^2$. Define an order on $\{\mathbf{e}_1, \mathbf{e}_2\}$ by $\mathbf{e}_1 < \mathbf{e}_2$. Then*

1. *For each $(n, k) \in \mathbb{N} \times \mathbb{N}$, $\mathbf{e}_1^{\otimes n} \otimes \mathbf{e}_2^{\otimes k}$ is a Lyndon word in the free Lie algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2\}$ with respect to the tensor product.*
2. *For each $n, k \in \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$, let $\mathcal{P}_{\mathbf{e}_1^{\otimes(n+1)} \otimes \mathbf{e}_2^{\otimes(k+1)}}$ be the Lyndon element corresponding to the Lyndon word $\mathbf{e}_1^{\otimes(n+1)} \otimes \mathbf{e}_2^{\otimes(k+1)}$. Then, for all $n, k \in \mathbb{N} \cup \{0\} \times$*