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Wave front set for solutions to Schrödinger equations with long-range perturbed harmonic oscillators

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ABSTRACT

In this paper we consider a Schrödinger operator with variable coefficients and harmonic potential. The perturbation is assumed to be long-range in a sense similar to the work of Nakamura (2009) [13]. We construct a modified propagator, and then by using this propagator and also the propagator of the unperturbed free harmonic oscillator we characterize the propagation of singularities for solutions to the equations.

1. Introduction

In this paper we consider a Schrödinger operator with variable coefficients and the harmonic potential:

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$$H = -\frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j} a_{jk}(x)\partial_{x_k} + \frac{1}{2}|x|^2 + V(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, $n \geqslant 1$. We denote the unperturbed free harmonic oscillator by H_0:

$$H_0 = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 \quad \text{on } \mathcal{H},$$

and we suppose H is a long-range perturbation of H_0 in the following sense:

Assumption A. $a_{jk}(x), V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ for $j, k = 1, \dots, n$, and $(a_{jk}(x))_{j,k}$ is positive symmetric for each $x \in \mathbb{R}^n$. Moreover, there exists $\mu > 0$ such that for any $\alpha \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha \big(a_{jk}(x) - \delta_{jk}\big)\right| \leqslant C_\alpha \langle x\rangle^{-\mu-|\alpha|},$$
$$\left|\partial_x^\alpha V(x)\right| \leqslant C_\alpha \langle x\rangle^{2-\mu-|\alpha|}$$

for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$, where $\langle x\rangle = \sqrt{1+|x|^2}$.

Then it is well-known that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, and we denote the unique self-adjoint extension by the same symbol H. We denote the symbols of H, H_0, the kinetic energy and the free Schrödinger operator by p, p_0, k and k_0, respectively. Namely, we denote

$$p(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + \frac{1}{2}|x|^2 + V(x),$$
$$p_0(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2,$$
$$k(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k, \qquad k_0(x,\xi) = \frac{1}{2}|\xi|^2.$$

We denote the Hamilton flow generated by a symbol $a(x,\xi)$ on $\mathbb{R}^{2n}$ by $\exp(tH_a) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, i.e., $\exp(tH_a)(x_0,\xi_0) := (x(t),\xi(t))$ denotes the solution to the following ODE:

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{\partial a}{\partial \xi}\big(x(t),\xi(t)\big), \\ \dfrac{d\xi(t)}{dt} = -\dfrac{\partial a}{\partial x}\big(x(t),\xi(t)\big), \\ x(0) = x_0, \qquad \xi(0) = \xi_0. \end{cases}$$

Then we can denote

$$\ell(t,x,\xi) = p \circ \exp(tH_{p_0})(x,\xi) - p_0(x,\xi),$$