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An elliptic semilinear equation with source term and boundary measure data: The supercritical case

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ABSTRACT

We give new criteria for the existence of weak solutions to an equation with a super linear source term

$$-\Delta u = u^q \quad \text{in } \Omega, \quad u = \sigma \quad \text{on } \partial\Omega$$

where Ω is either a bounded smooth domain or $\mathbb{R}_+^N$, $q > 1$ and $\sigma \in \mathfrak{M}^+(\partial\Omega)$ is a nonnegative Radon measure on $\partial\Omega$. One of the criteria we obtain is expressed in terms of some Bessel capacities on $\partial\Omega$. We also give a sufficient condition for the existence of weak solutions to equation with source mixed terms.

$$-\Delta u = |u|^{q_1-1}u|\nabla u|^{q_2} \quad \text{in } \Omega, \quad u = \sigma \quad \text{on } \partial\Omega$$

where $q_1, q_2 \geq 0$, $q_1 + q_2 > 1$, $q_2 < 2$, $\sigma \in \mathfrak{M}(\partial\Omega)$ is a Radon measure on $\partial\Omega$.

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1. Introduction and main results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty)$, $N \geq 3$, and $g : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function. In this paper, we study the solvability of the problem

$$\begin{aligned} -\Delta u &= g(u, \nabla u) && \text{in } \Omega, \\ u &= \sigma && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\sigma \in \mathfrak{M}(\partial\Omega)$ is a Radon measure on $\partial\Omega$. All solutions are understood in the usual very weak sense, which means that $u \in L^1(\Omega)$, $g(u, \nabla u) \in L^1_\rho(\Omega)$, where $\rho(x)$ is the distance from x to $\partial\Omega$ when Ω is bounded, or $u \in L^1(\mathbb{R}^N_+ \cap B)$, $g(u, \nabla u) \in L^1_\rho(\mathbb{R}^N_+ \cap B)$ for any ball B if $\Omega = \mathbb{R}^N_+$, and

$$\int_\Omega u(-\Delta\xi)dx = \int_\Omega g(u, \nabla u)\xi dx - \int_{\partial\Omega} \frac{\partial\xi}{\partial n} d\sigma \tag{1.2}$$

for any $\xi \in C^2(\overline{\Omega}) \cap C_c(\mathbb{R}^N)$ with $\xi = 0$ in Ω^c, where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$, n is the outward unit vector on $\partial\Omega$. It is well known that such a solution u satisfies

$$u = \mathbf{G}[g(u, \nabla u)] + \mathbf{P}[\sigma] \quad \text{a.e. in } \Omega,$$

where $\mathbf{G}[.]$, $\mathbf{P}[.]$, respectively the Green and the Poisson potentials associated with $-\Delta$ in Ω, are defined from the Green and the Poisson kernels by

$$\mathbf{P}[\sigma](y) = \int_{\partial\Omega} \mathrm{P}(y, z)d\sigma(z), \quad \mathbf{G}[g(u, \nabla u)](y) = \int_\Omega \mathrm{G}(y, x)g(u, \nabla u)(x)dx,$$

see [16].

Our main goal is to establish necessary and sufficient conditions for the existence of weak solutions of (1.1) with boundary measure data, together with sharp pointwise estimates of the solutions. In the sequel we study two cases for the problem (1.1):

1. The pure power case

$$\begin{aligned} -\Delta u &= |u|^{q-1}u && \text{in } \Omega, \\ u &= \sigma && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

with $u \geq 0$, $q > 1$ and $\sigma \geq 0$.