Journal of Functional Analysis 269 (2015) 2018–2044

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Geometric duality theory of cones in dual pairs of vector spaces

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ARTICLE INFO

Article history:
Received 2 January 2015
Accepted 26 April 2015
Available online 8 May 2015
Communicated by G. Schechtman

MSC:
primary 46A20
secondary 46B10, 46B20, 46A40, 46B40, 46L05

Keywords:
Cone
Duality theory
Banach space
C*-algebra

ABSTRACT

This paper will generalize what may be termed the "geometric duality theory" of real pre-ordered Banach spaces which relates geometric properties of a closed cone in a real Banach space, to geometric properties of the dual cone in the dual Banach space. We show that geometric duality theory is not restricted to real pre-ordered Banach spaces, as is done classically, but can be extended to real Banach spaces endowed with arbitrary collections of closed cones.

We define geometric notions of normality, conormality, additivity and coadditivity for members of dual pairs of real vector spaces as certain possible interactions between two cones and two convex sets containing zero. We show that, thus defined, these notions are dual to each other under certain conditions, i.e., for a dual pair of real vector spaces (Y, Z), the space Y is normal (additive) if and only if its dual Z is conormal (coadditive) and vice versa. These results are set up in a manner so as to provide a framework to prove results in the geometric duality theory of cones in real Banach spaces. As an example of using this framework, we generalize classical duality results for real Banach spaces pre-ordered by a single closed cone, to real Banach spaces endowed with an arbitrary collections of closed cones.

As an application, we analyze some of the geometric properties of naturally occurring cones in C*-algebras and their duals.

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http://dx.doi.org/10.1016/j.jfa.2015.04.022

1. Introduction

The goal of this paper is to provide a general framework for proving results in what may be termed the "geometric duality theory" of cones in real Banach spaces. We will prove the General Duality Theorems (Theorems 3.4 and 3.5) for dual pairs of real vector spaces and, as an application, we generalize classical results for real pre-ordered Banach spaces (cf. Theorems 1.2 and 1.3) to the context of real Banach spaces endowed with arbitrary collections of closed cones (cf. Corollaries 4.7 and 4.9).

We begin with some motivating historical remarks:

Andô's Theorem [2, Lemma 1], a fundamental result in the geometric theory of real pre-ordered Banach spaces, states:

Theorem 1.1. *Let X be a real Banach space, pre-ordered by a closed cone X_+ (in the sense that $x \leq y$ means $y \in x + X_+$). If the cone X_+ generates X, i.e., $X = X_+ - X_+$, then there exists a constant $\alpha \geq 1$, such that every $x \in X$ can be written as $x = a - b$ with $a, b \in X_+$ and $\max\{\|a\|, \|b\|\} \leq \alpha \|x\|$.*

The geometric property given by the conclusion of Andô's Theorem is what is termed a 'conormality property[1]' (cf. property (2)(a) in Theorem 1.2 for another example).

It has long been known that there exists a geometric duality theory for such relationships between norms and closed cones in real pre-ordered Banach spaces. Loosely speaking, there is a dual notion to conormality, called 'normality[2]' (cf. property (1)(a) in Theorem 1.2 for a specific example of a normality property). Normality is 'dual' to conormality in the sense that a real pre-ordered Banach space has a conormality property if and only if its dual has a corresponding normality property, and vice versa. Many such dual pairs of normality and conormality properties have been discovered for real pre-ordered Banach spaces. A fairly complete inventory of normality and conormality properties and their relationships as dual properties is given with full references in [16, Definition 3.1, Theorem 3.7].

The following is a representative sample of duality results in this vein. To the author's knowledge, first proofs of this particular result date back to Grosberg and Krein [11], and Ellis [10]:

Theorem 1.2. *Let $\alpha \geq 1$. Let X be real Banach space, pre-ordered by a closed cone X_+. Let the dual space X' be pre-ordered by the dual cone $X'_+ := \{\phi \in X' \mid \phi(X_+) \subseteq \mathbb{R}_{\geq 0}\}$, where $\mathbb{R}_{\geq 0} := \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$.*

[1] The term 'conormality' is due to Walsh [19]. Historically this notion appears under various names in the literature, of which 'α-generating' or 'boundedly generating' is quite common.

[2] The term 'normality' is due to Krein [15].