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Martingale inequalities in noncommutative symmetric spaces

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ABSTRACT

We provide generalizations of Burkholder's inequalities involving conditioned square functions of martingales to the general context of martingales in noncommutative symmetric spaces. More precisely, we prove that Burkholder's inequalities are valid for any martingale in noncommutative space constructed from a symmetric space defined on the interval $(0, \infty)$ with the Fatou property and whose Boyd indices are strictly between 1 and 2. This answers positively a question raised by Jiao and may be viewed as a conditioned version of similar inequalities for square functions of noncommutative martingales. Using duality, we also recover the previously known case where the Boyd indices are finite and are strictly larger than 2.

1. Introduction

In classical martingale theory, a fundamental result due to Burkholder [8,9,20] can be described as follows: given a probability space (Ω, Σ, P), let $\{\Sigma_n\}_{n\geq 1}$ be an increasing

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sequence of σ-fields of Σ such that $\Sigma = \bigvee \Sigma_n$. If $2 \leq p < \infty$ and $f = (f_n)_{n\geq 1}$ is an L_p-bounded martingale adapted to the filtration $\{\Sigma_n\}_{n\geq 1}$, then (using the convention that $\Sigma_0 = \Sigma_1$),

$$\sup_{n\geq 1}[\mathbb{E}|f_n|^p]^{1/p} \simeq_p \Big[\mathbb{E}\big(\sum_{n\geq 1}\mathbb{E}[|df_n|^2|\Sigma_{n-1}]\big)^{p/2}\Big]^{1/p} + \Big[\sum_{n\geq 1}\mathbb{E}|df_n|^p\Big]^{1/p}, \tag{1.1}$$

where $\simeq_p$ means equivalence of norms up to constants depending only on p. The random variable $s(f) = \big(\sum_{n\geq 1}\mathbb{E}[|df_n|^2|\Sigma_{n-1}]\big)^{1/2}$ is called the conditioned square function of the martingale f and the equivalence (1.1) is generally referred to as Burkholder's inequalities. The equivalence (1.1) was established by Burkholder as the martingale difference sequence generalizations of Rosenthal's inequalities [44] which state that if $2 \leq p < \infty$ and $(g_n)_{n\geq 1}$ is a sequence of independent mean-zero random variables in $L_p(\Omega, \Sigma, P)$ then

$$\Big(\mathbb{E}\big|\sum_{n\geq 1} g_n\big|^p\Big)^{1/p} \simeq_p \Big(\sum_{n\geq 1}\mathbb{E}|g_n|^2\Big)^{1/2} + \Big(\sum_{n\geq 1}\mathbb{E}|g_n|^p\Big)^{1/p}. \tag{1.2}$$

Probabilistic inequalities involving independent random variables and martingales inequalities play important roles in many different areas of mathematics. The Burkholder/Rosenthal inequalities in particular have many applications in probability theory and structures of symmetric spaces in Banach space theory. On the other hand, a recent trend in the general study of martingale inequalities is to find analogues of classical inequalities in the context of noncommutative L_p-spaces. We refer to [39,26,28,41] for additional information on noncommutative martingale inequalities. Noncommutative analogues of (1.1) and (1.2) were extensively studied by Junge and Xu in [29,30]. They obtained that if $2 \leq p < \infty$ and $x = (x_n)_{n\geq 1}$ is a noncommutative martingale that is L_p-bounded then

$$\|x\|_p \simeq_p \max\Big\{\|s_c(x)\|_p, \|s_r(x)\|_p, \big(\sum_{n\geq 1}\|dx_n\|_p^p\big)^{1/p}\Big\} \tag{1.3}$$

where $s_c(x)$ and $s_r(x)$ denote the column version and the row version of conditioned square functions which we refer to the next section for formal definitions. Moreover, they also treated the corresponding inequalities for the range $1 < p < 2$ which are dual versions of (1.3) and read as follows: if $x = (x_n)_{n\geq 1}$ is a noncommutative martingale in $L_2(\mathcal{M})$ then

$$\|x\|_p \simeq_p \inf\Big\{\|s_c(y)\|_p + \|s_r(z)\|_p + \big(\sum_{n\geq 1}\|dw_n\|_p^p\big)^{1/p}\Big\} \tag{1.4}$$

where the infimum is taken over all $x = y + z + w$ with y, z, and w are martingales. The differences between the two cases $1 < p < 2$ and $2 \leq p < \infty$ are now well-understood in the field. In [21], inequalities (1.3) and (1.4) were extended to the case of noncommutative