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Harnack estimates for nonlinear backward heat equations in geometric flows

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ABSTRACT

Let M be a closed Riemannian manifold with a family of Riemannian metrics $g_{ij}(t)$ evolving by a geometric flow $\partial_t g_{ij} = -2S_{ij}$, where $S_{ij}(t)$ is a family of smooth symmetric two-tensors. We derive several differential Harnack estimates for positive solutions to the nonlinear backward heat-type equation

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf$$

where a and γ are constants and $S = g^{ij} S_{ij}$ is the trace of S_{ij}. Our abstract formulation provides a unified framework for some known results proved by various authors, and moreover leads to new Harnack inequalities for a variety of geometric flows.

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1. Introduction

The study of differential Harnack estimates for parabolic equations originated with the celebrated works [1,17]. In 1979, Aronson and Bénilan [1] derived gradient estimates for the heat and porous medium equation in $\mathbb{R}^n$. On the other hand, Li and Yau [17] proved a gradient estimate for the heat equation on a complete Riemannian manifold by using the maximum principle. By integrating the gradient estimate along a space-time path, a classical Harnack inequality was derived. Therefore, Li–Yau type gradient estimate is often called differential Harnack estimate. Similar techniques were used by Hamilton to prove Harnack estimates for the Ricci flow [11,12] and the mean curvature flow [14].

Using similar techniques, many authors have proved a variety of Li–Yau–Hamilton's Harnack estimates for various equations in different geometric flows, and we refer to the survey paper by Ni [23]. In the Ricci flow, Perelman proved a Harnack inequality for the fundamental solution of the conjugate heat equation [24] and later on different Harnack inequalities have been proved, to name but a few [2–4,16,19,24,26,27]. Harnack inequalities for more general flows have been considered in [5–7,15,25].

The main purpose of the current article is, in the framework of a general geometric flow, to derive Li–Yau–Hamilton type differential Harnack estimates for positive solutions to a nonlinear backward heat-type equation generalizing Perelman's conjugate heat equation.

Let M be a closed Riemannian n-manifold with a one parameter family of Riemannian metrics $g(t)$ evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \quad t \in [0, T) \tag{1}$$

where $S_{ij}(t)$ is any smooth symmetric two-tensor on $(M, g(t))$. f is a positive solution to

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf \tag{2}$$

where symbol Δ stands for the Laplacian of the evolving metric $g(t)$, γ and a are constants and $S = g^{ij} S_{ij}$ is the trace of S_{ij}. In the Ricci flow case, when $\gamma = 0$ and $a = 1$, (2) is the conjugate heat equation introduced by Perelman. The consideration of this nonlinear equation is motivated by gradient Ricci solitons. See [4,26] for more details. In a forthcoming paper [8], we will consider the forward nonlinear heat-type equation.

To state the main results, we introduce evolving tensor quantities associated to the tensor S_{ij}.

Definition 1. Let $g(t)$ evolve by (1) and $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on M. For a constant $a \in \mathbb{R}$, we define

$$\mathcal{E}_a(S_{ij}, X) = \left(a\frac{\partial S}{\partial \tau} + a\Delta S + 2|S_{ij}|^2 \right) - 2\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right) X^\ell - 2\left(R^{ij} - S^{ij}\right) X_i X_j, \tag{3}$$