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On the Cauchy problem for the two-component Euler–Poincaré equations

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ABSTRACT

In the paper, we first use the energy method to establish the local well-posedness as well as blow-up criteria for the Cauchy problem on the two-component Euler–Poincaré equations in multi-dimensional space. In the case of dimensions 2 and 3, we show that for a large class of smooth initial data with some concentration property, the corresponding solutions blow up in finite time by using Constantin–Escher Lemma and Littlewood–Paley decomposition theory. Then for the one-component case, a more precise blow-up estimate and a global existence result are also established by using similar methods. Next, we investigate the zero density limit and the zero dispersion limit. At the end, we also briefly demonstrate a Liouville type theorem for the stationary weak solution.

Contents

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1. Introduction

In this paper, we consider the Cauchy problem on the following two-component Euler–Poincaré equations in multi-dimensional space $\mathbb{R}^N$ $(N \geq 2)$:

$$\begin{cases} m_t + u \cdot \nabla m + (\nabla u)^T m + m \nabla \cdot u = -\rho \nabla \rho, & \text{in } \mathbb{R}^N \times (0,T), \\ \rho_t + \nabla \cdot (\rho u) = 0, & \text{in } \mathbb{R}^N \times (0,T), \\ m = \left(1 - \alpha^2 \Delta\right) u, & \text{in } \mathbb{R}^N \times (0,T), \\ m(x,0) = m_0(x), \quad \rho(x,0) = \rho_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, \cdots, u_N)$ represents the velocity of fluid, $m = (m_1, m_2, \cdots, m_N)$ denotes the momentum, and the scalar function ρ stands for the density or the total depth. The notation $(\nabla u)^T$ denotes the transpose of the matrix ∇u. The constant $\alpha > 0$ corresponds to the length scale and is called the dispersion parameter. Eqs. (1.1) were presented by [22,25] as a framework for modeling and analyzing fluid dynamics, particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling, or recasting the geodesic flow on the diffeomorphism groups. In the case of $\alpha = 0$, Eqs. (1.1) are called zero-dispersive Euler–Poincaré equations and can be written as

$$\begin{cases} u_t + u \cdot \nabla u + (\nabla u)^T u + u \nabla \cdot u = -\rho \nabla \rho, & \text{in } \mathbb{R}^N \times (0,T), \\ \rho_t + \nabla \cdot (\rho u) = 0, & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

which is a symmetric hyperbolic system of conservation laws (see (2.5) below).

To motivate our study, we recall some related progresses on Eqs. (1.1). When the system is decoupled (i.e., formally, $\rho \equiv 0$), Eqs. (1.1) reduce to the classical mathematical model of the fully nonlinear shallow water waves or the one of the geodesic motion on diffeomorphism group:

$$\begin{cases} m_t + u \cdot \nabla m + (\nabla u)^T m + m \nabla \cdot u = 0, & \text{in } \mathbb{R}^N \times (0,T), \\ m = \left(1 - \alpha^2 \Delta\right) u, & \text{in } \mathbb{R}^N \times (0,T), \\ m(x,0) = m_0(x), & \text{in } \mathbb{R}^N \end{cases} \tag{1.3}$$

(see [3,5,20,21,23]). In particular, Eqs. (1.3) are the classical Camassa–Holm equations for $N = 1$, while they are also called the Euler–Poincaré equations in the higher dimensional case $N \geq 1$. The local well-posedness, blow-up criterion, existence of blow-up