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Measure-theoretic degrees and topological pressure for non-expanding transformations

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ABSTRACT

We consider invariant sets Λ of saddle type, for non-invertible smooth maps f, and equilibrium measures μ_ϕ associated to Hölder potentials ϕ on Λ. We define a notion of *measure-theoretic asymptotic degree of* $f|_\Lambda : \Lambda \to \Lambda$, with respect to the measure μ_ϕ on the fractal set Λ. In our case, the equilibrium measure μ_ϕ is the unique linear functional in $\mathcal{C}(\Lambda)^*$ tangent to the pressure function $P(\cdot) : \mathcal{C}(\Lambda) \to \mathbb{R}$ at ϕ. In particular, for the measure of maximal entropy μ_0 of $f|_\Lambda$, we obtain the *asymptotic degree* of $f|_\Lambda$, which represents the average rate of growth of the *number of n-preimages of x that remain* in Λ when $n \to \infty$; notice that, in general, Λ is not totally invariant for f. To this end, we will obtain first a formula for the Jacobians of the probability μ_ϕ, with respect to arbitrary iterates f^m, $m \geq 2$. We then show that a formula for the topological pressure $P(\phi)$ that holds in the expanding case, is no longer true on saddle sets. In the saddle case we find a *new formula for the pressure*, involving weighted sums on preimage sets. We also apply the asymptotic degrees, together with various pressure functionals, in order to obtain estimates for the Hausdorff dimension of stable slices through certain sets of full μ_ϕ-measure in the fractal Λ. In the end, we give also some concrete examples on saddle folded sets.

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1. Introduction and outline of main results

We consider smooth maps $f : M \to M$ on a manifold M, which are hyperbolic and non-invertible on saddle locally maximal sets Λ, and associated equilibrium (Gibbs) measures μ_ϕ, of Hölder potentials ϕ on Λ. We investigate several notions related to them, like the Jacobian of such a measure, and a new, measure-theoretic notion of "degree" of $f|_\Lambda : \Lambda \to \Lambda$, in the case when the number of f-preimages that remain in Λ of an arbitrary point x, is not constant, when x varies in Λ. We will also look more closely at the pressure functional $P(\cdot)$, on the Banach space $\mathcal{C}(\Lambda)$ of continuous real-valued functions on Λ, when Λ is such a saddle non-invertible fractal set. For background on invariant sets for dynamical systems, equilibrium measures, pressure, etc., see for e.g. [2,4–6,15,20,23,27].

The hyperbolic non-expanding and non-invertible case is very different from the expanding case, and from the hyperbolic diffeomorphism case (e.g. [5,20,10]). One difficulty is that branches of inverse iterates do not contract small balls on Λ, which means that the machinery from the expanding case cannot be used here; in fact inverse branches dilate on stable directions. Another difficulty is that, as the fractal set Λ is not necessarily totally invariant with respect to f, there may be sudden variations in the number of f-preimages in Λ of a point x, when x ranges in Λ; also, there exist in general many (possibly uncountably) local unstable manifolds through points in Λ.

We will obtain first a formula for the Jacobian of the equilibrium measure μ_ϕ with respect to an arbitrary iterate f^m, $m \geq 2$, in this saddle case (the Jacobian is in fact a Radon–Nikodym derivative). Using this, we obtain then a formula for the pressure $P(\phi)$ in terms of the preimage sets of μ_ϕ-almost all points x, and of the folding entropy of μ_ϕ. This formula is different from the one in the expanding case (e.g. from [17]).

In general, the map f is not constant-to-1 on the saddle fractal Λ. Thus, we want to determine a good notion of "degree" for the restriction $f|_\Lambda$. We find one such notion of asymptotic degree with respect to the measure μ_ϕ. If we consider in particular the measure of maximal entropy μ_0 on Λ, we obtain then the average logarithmic growth of the number of n-preimages that remain in Λ (when $n \to \infty$), which can be considered as the "degree" of f over Λ. By using the above notions of asymptotic degree with respect to μ_ϕ, we will obtain next estimates for the dimension of stable slices through certain explicitly constructed sets of full μ_ϕ-measure in Λ.

Hence, the asymptotic degrees, the formula for Jacobians of equilibrium states with respect to arbitrary iterates and the associated methods, are useful in obtaining:

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