Journal of Functional Analysis 267 (2014) 2846–2883

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Multi-dimensional scalar balance laws with discontinuous flux

Piotr Gwiazda [a], Agnieszka Świerczewska-Gwiazda [a,*], Petra Wittbold [b], Aleksandra Zimmermann [b]

[a] *Institute of Applied Mathematics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland*
[b] *Faculty of Mathematics, University of Duisburg-Essen, 45117 Essen, Germany*

A R T I C L E I N F O

Article history:
Received 17 December 2013
Accepted 10 July 2014
Available online 5 August 2014
Communicated by C. De Lellis

MSC:
35L65
35R05

Keywords:
Entropy weak solutions
Entropy measure-valued solutions
Semi-Kružkov entropies
Discontinuous flux

A B S T R A C T

We consider scalar balance laws with a dissipative source term. The flux function may be discontinuous with respect to both the space variable x and the unknown quantity u. We formulate the definition of entropy weak solutions and provide existence and uniqueness to the considered problem. The problem is formulated in the framework of multi-valued mappings. The notion of entropy measure-valued solutions is used to prove the so-called contraction principle and comparison principle.

* Corresponding author.
E-mail addresses: pgwiazda@mimuw.edu.pl (P. Gwiazda), aswiercz@mimuw.edu.pl (A. Świerczewska-Gwiazda), petra.wittbold@uni-due.de (P. Wittbold), aleksandra.zimmermann@uni-due.de (A. Zimmermann).

http://dx.doi.org/10.1016/j.jfa.2014.07.009
0022-1236/

1. Introduction

Our interest is directed to the following Cauchy problem describing the evolution of $u : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$

$$u_t + \operatorname{div} \Phi(x, u) \ni f(t, x, u) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^N, \tag{1.1}$$

$$u(0, \cdot) = u_0 \quad \text{on } \mathbb{R}^N. \tag{1.2}$$

where $\Phi : \mathbb{R}^N \times \mathbb{R} \to 2^{\mathbb{R}^N}$ is a multi-valued mapping and $f : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a source term. Moreover $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given initial condition in $L^\infty(\mathbb{R}^N)$. The assumptions for Φ and f will be presented below. The formulation of the problem in the language of multi-valued flux function allows to capture relations which are not necessarily functions.

We will assume that the flux function is in the form of a composition, which allows, with an appropriate change of variables, to formulate the definition of entropy weak solutions in terms of the new variables. An important property of solutions defined this way is that in case of smooth fluxes they correspond to the classical definition of entropy weak solutions, see e.g. Kružkov [25]. We assume about Φ and f that:

(H1) $\Phi(x, u)$ is a multi-valued mapping given by the formula $\Phi(x, u) = A(\beta(x, u))$ where $A : \mathbb{R} \to \mathbb{R}^N$, A is continuous and $\beta : \mathbb{R}^N \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a multi-valued mapping such that, for almost every $x \in \mathbb{R}^N$, $\beta(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a maximal monotone operator with $0 \in \beta(x, 0)$. We assume that the inverse to β (w.r.t. u), which we call α, is continuous. Moreover, we assume that $\beta^*(\cdot, l)$ is measurable for each $l \in \mathbb{R}$, where β^* denotes the minimal selection of the graph of β,

(H2) there exist continuous functions h_1 and h_2 with $\lim_{|u|\to\infty} h_1(u) = \infty$ such that

$$h_1(u) \le |\overline{\beta}| \le h_2(u) \tag{1.3}$$

for all $\overline{\beta} \in \beta(x, u)$, almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$,

(H3) there exist $1 \le p \le \frac{N}{N-1}$ and constants $R_\infty > 0$ and $C_\infty > 0$ such that for all $|x| > R_\infty$

$$\left|A(s)\right|^p \le C_\infty \left|\alpha(x, s)\right|,$$

(H4) $f(\cdot, \cdot, u) \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^N)$ for all $u \in \mathbb{R}$; $f(t, x, \cdot)$ is continuous and $f(t, x, 0) = 0$ for a.a. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Moreover f is dissipative ($-f$ is monotone w.r.t. the last variable), i.e.,

$$\big(f(t, x, u) - f(t, x, v)\big)(u - v) \le 0$$
$$\text{for all } u, v \in \mathbb{R} \text{ and a.a. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N. \tag{1.4}$$