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Classification of invariant valuations on the quaternionic plane [☆]

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ABSTRACT

We describe the orbit space of the action of the group $\mathrm{Sp}(2)\,\mathrm{Sp}(1)$ on the real Grassmann manifolds $\mathrm{Gr}_k(\mathbb{H}^2)$ in terms of certain quaternionic matrices of Moore rank not larger than 2. We then give a complete classification of valuations on the quaternionic plane $\mathbb{H}^2$ which are invariant under the action of the group $\mathrm{Sp}(2)\,\mathrm{Sp}(1)$.

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1. Introduction and statement of results

1.1. Background

A valuation is a finitely additive map from the space of compact convex subsets of some vector space to an abelian semi-group. Since Hadwiger's famous characterization of (real-valued) continuous valuations which are euclidean motion invariant, classification results for valuations have long played a prominent role in convex and integral geometry.

Many generalizations of Hadwiger's theorem were obtained recently. On the one hand, valuations with values in some abelian semi-group other than the reals were characterized. The most important examples are tensor valuations [5,19,20,28], Minkowski valuations [1,2,18,25,35,36], curvature measures [16,34] and area measures [42,43]. On the other hand, invariance with respect to the euclidean group was weakened to invariance with respect to translations or rotations only [4,6], or with respect to a smaller group of isometries. Next we briefly describe the main results in this line.

Let V be a finite-dimensional vector space and G a group acting linearly on V. The space of scalar-valued, G-invariant, translation invariant continuous valuations on V will be denoted by Val^G. Hadwiger's theorem applies in the case where V is a euclidean vector space of dimension n, and $G = \mathrm{SO}(V)$. It states that Val^G is spanned by the so-called intrinsic volumes $\mu_0, \ldots, \mu_n$. In particular, $\mathrm{Val}^{\mathrm{SO}(V)}$ is finite-dimensional. From this fact, one can easily derive integral-geometric formulas like Crofton formulas and kinematic formulas [24].

In the same spirit, kinematic formulas with respect to a smaller group G exist provided that Val^G is finite-dimensional. Although it is known which groups have this property, much less is known about the explicit form of such formulas. Alesker [10] has shown that Val^G is finite-dimensional if and only if G acts transitively on the unit sphere. Such groups were classified by Montgomery and Samelson [29] and Borel [17]. There are six infinite lists

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n)\mathrm{U}(1), \mathrm{Sp}(n)\,\mathrm{Sp}(1) \tag{1}$$

and three exceptional groups

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9). \tag{2}$$

The euclidean case is $G = \mathrm{SO}(n)$ where Hadwiger's theorem applies. In the hermitian case $G = \mathrm{U}(n)$ or $G = \mathrm{SU}(n)$, recent results have revealed a lot of unexpected algebraic structures yielding a relatively complete picture [3,6,15,16,33,39]. Hadwiger-type theorems for the groups G_2 and $\mathrm{Spin}(7)$ are also known [13]. In the remaining cases, i.e. the quaternionic cases $G = \mathrm{Sp}(n)$, $G = \mathrm{Sp}(n)\mathrm{U}(1)$ and $G = \mathrm{Sp}(n)\,\mathrm{Sp}(1)$ as well as in the case $G = \mathrm{Spin}(9)$, only the dimension of Val^G is known [14,41].

The combinatorial formulas from [14] indicate that the classification of invariant valuations on quaternionic vector spaces will be a rather subtle subject. Note that the case