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Sharp results for the Weyl product on modulation spaces

Elena Cordero [a], Joachim Toft [b,*], Patrik Wahlberg [b]

[a] *Department of Mathematics, University of Turin, Via Carlo Alberto 10, 10123 Torino (TO), Italy*
[b] *Department of Mathematics, Linnæus University, Växjö, Sweden*

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ABSTRACT

We give sufficient and necessary conditions on the Lebesgue exponents for the Weyl product to be bounded on modulation spaces. The sufficient conditions are obtained as the restriction to $N = 2$ of a result valid for the N-fold Weyl product. As a byproduct, we obtain sharp conditions for the twisted convolution to be bounded on Wiener amalgam spaces.

0. Introduction

In the paper we prove necessary and sufficient conditions for the Weyl product to be continuous on modulation spaces, and for the twisted convolution to be continuous on

* Corresponding author.
E-mail addresses: elena.cordero@unito.it (E. Cordero), joachim.toft@lnu.se (J. Toft), patrik.wahlberg@lnu.se (P. Wahlberg).

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Wiener amalgam spaces. We relax the sufficient conditions in [27] and we prove that the obtained conditions are also necessary.

The Weyl calculus is a part of the theory of pseudo-differential operators. For an appropriate distribution a (the symbol) defined on the phase space $T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$, the *Weyl operator* $\mathrm{Op}^w(a)$ is a linear map between spaces of functions or distributions on $\mathbf{R}^d$. (See Section 1 for definitions.) Weyl operators appear in various fields. In mathematical analysis they are used to represent linear operators, in particular linear partial differential operators, acting between appropriate function and distribution spaces. Weyl operators also appear in quantum mechanics where a real-valued observable a in classical mechanics corresponds to the self-adjoint Weyl operator $\mathrm{Op}^w(a)$ in quantum mechanics. For this reason $\mathrm{Op}^w(a)$ is often called the Weyl quantization of a. In time-frequency analysis pseudo-differential operators are used as models of non-stationary filters.

In the Weyl calculus operator composition corresponds on the symbol level to the *Weyl product*, or the *twisted product*, denoted by #. This means that the Weyl product $a_1 \# a_2$ of appropriate functions or distributions a_1 and a_2 satisfies

$$\mathrm{Op}^w(a_1 \# a_2) = \mathrm{Op}^w(a_1) \circ \mathrm{Op}^w(a_2).$$

A basic problem is to find conditions that are necessary or sufficient for the bilinear map

$$(a_1, a_2) \mapsto a_1 \# a_2 \tag{0.1}$$

to be well-defined and continuous. Here we investigate these questions when the factors belong to modulation spaces, a family of Banach spaces of distributions which appear in time-frequency analysis, harmonic analysis and Gabor analysis.

The modulation spaces were introduced by Feichtinger [6], and their theory was further developed and generalized by Feichtinger and Gröchenig [8–10,15] into the theory of coorbit spaces.

The modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, where $p, q \in [1, \infty]$ and ω is a weight on $\mathbf{R}^d \times \mathbf{R}^d \simeq \mathbf{R}^{2d}$, consists of all tempered distributions, or ultra-distributions, on $\mathbf{R}^d$, whose short-time Fourier transforms have finite $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ norm. Thus the Lebesgue exponents p and q, and above all the weight ω, give a scale of function spaces $M^{p,q}_{(\omega)}$ with respect to phase space concentration. The definition of modulation spaces resembles that of Besov spaces, and narrow embeddings between modulation and Besov spaces have been found (cf. [14,26,32,38,44,46,50,52]). Depending on the assumptions on the weights, the modulation spaces are subspaces of the tempered distributions or ultra-distributions (cf. [2,34,35,40,48,49]).

Since the early 1990s modulation spaces have been used in the theory of pseudo-differential operators (cf. [39]). Sjöstrand [36] introduced the modulation space $M^{\infty,1}(\mathbf{R}^{2d})$, which contains non-smooth functions, as a symbol class. He proved that $M^{\infty,1}$ corresponds to an algebra of L^2-bounded operators.

Gröchenig and Heil [20,16] proved that each operator with symbol in $M^{\infty,1}$ is continuous on all modulation spaces $M^{p,q}$, $p, q \in [1, \infty]$. This extends Sjöstrand's L^2-continuity