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A new approach to spectral approximation

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ABSTRACT

A new technique for approximating eigenvalues and eigenvectors of a self-adjoint operator is presented. The method does not incur spectral pollution, uses trial spaces from the form domain, has a self-adjoint algorithm, and exhibits superconvergence.

1. Introduction

We shall introduce a new technique for computing eigenvalues and eigenvectors of an arbitrary semi-bounded self-adjoint operator. The method can reliably locate those eigenvalues which, due to *spectral pollution*, are not successfully approximated by a direct application of the Galerkin (finite section) method; see for example [1,3,4,7,9,11,13,20,23]. The technique is very easy to apply, uses trial spaces from the form domain, has a self-adjoint algorithm, and exhibits the superconvergence of the Galerkin method. The

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only comparable technique is the recently developed perturbation method; see [21,22,28,29]. However, the latter involves perturbing eigenvalues off the real line and then approximating them, consequently, the method is compromised by having a non-self-adjoint algorithm.

There have been two approaches to locating eigenvalues and eigenvectors when spectral pollution precludes approximation with the Galerkin method. Firstly, certain specialised techniques have been proposed, with each being applicable only to a particular class of differential operator; see for example [1,7,14,21,22,30–32]. Secondly, general techniques which are applicable to self-adjoint or even to arbitrary operators; see for example [12,13,15–17,20,24,28,29,34]. The last two decades has seen an enormous effort directed at general techniques for self-adjoint operators. This effort led to the development of *quadratic* methods, so-called because of their reliance on truncations of the square of the operator. Although pollution-free, these techniques have significant drawbacks. They require trial spaces from the operator domain, rather than the preferred form domain. The latter being far more convenient, for example, it will usually contain the FEM space of piecewise linear trial functions. Furthermore, only the form domain is typically supported by FEM software, consequently, applying a quadratic method can be arduous. A more serious problem concerns convergence rates. Quadratic methods will typically converge to eigenvalues very slowly when compared to the superconvergence of the Galerkin method; see for example [27, Section 6], [9, Examples 3.5 & 4.3] and [29, Example 8]. The quadratic method which has received the most attention is the *second order relative spectrum*; see for example [5,6,8,9,12,20,24–27]. This technique has the additional drawback of a non-self-adjoint algorithm.

In Section 2, we establish the strong convergence, with respect to the appropriate norm, of Galerkin spectral projections which are associated to a given interval. In Section 3, we give and justify a new definition of spectral pollution. Our main result is Theorem 4.3 which shows that by applying the Galerkin method to an auxiliary spectral problem, with respect to certain Galerkin spectral subspaces, we achieve pollution-free spectral approximation for our underlying self-adjoint operator. In Section 5, we apply our new technique to self-adjoint operators whose eigenvalues are not located by a direct application of the Galerkin method.

2. Galerkin spectral projections

Throughout, A, denotes a bounded or semi-bounded (from below) self-adjoint operator acting on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. The corresponding quadratic form we denote by $\mathfrak{a}$. $\mathcal{H}_{\mathfrak{a}}$ will be the Hilbert space with the inner-product

$$\langle u,v\rangle_{\mathfrak{a}} := \mathfrak{a}[u,v] - (m-1)\langle u,v\rangle \quad \forall u,v\in \mathrm{Dom}(\mathfrak{a}) \text{ where } m := \min\sigma(A)$$

and norm

$$\|u\|_{\mathfrak{a}} = \big(\mathfrak{a}[u,u] - (m-1)\langle u,u\rangle\big)^{\frac{1}{2}} = \big\|(A-m+1)^{\frac{1}{2}}u\big\|.$$