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Uniform pointwise bounds for matrix coefficients of unitary representations on semidirect products

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A B S T R A C T

Let k be a local field of characteristic 0, and let G be a connected semisimple almost k-algebraic group. Suppose $\mathrm{rank}_k G \geqslant 1$ and ρ is an excellent representation of G on a finite dimensional k-vector space V. We construct uniform pointwise bounds for the K-finite matrix coefficients restricted on G of all unitary representations of the semi-direct product $G \ltimes_\rho V$ without non-trivial V-fixed vectors. These bounds turn out to be sharper than the bounds obtained from G itself for some cases. As an application, we discuss a simple method of calculating Kazhdan constants for various compact subsets of the pair $(G \ltimes_\rho V, V)$.

1. Introduction and main results

Let k be a local field of $\mathrm{char}\, k = 0$. We say that G is a (connected) *almost k-algebraic group* if G is a (connected) k-Lie group with finite center for k isomorphic to $\mathbb{R}$ or G is the group of k-rational points of a (connected) linear algebraic group $\tilde{G}$ over k for k

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non-archimedean or isomorphic to $\mathbb{C}$. Unless stated otherwise, G denotes a connected semisimple almost k-algebraic group with $\mathrm{rank}_k(G) \geqslant 1$ and $\tilde{G}$ denotes its underlying algebraic group; that is, $G = \tilde{G}(k)$ for k non-archimedean or isomorphic to $\mathbb{C}$.

1.1. Finite-dimensional representations of G

For a finite dimensional vector space V over k, a representation $\rho : G \to GL(V)$ is called *normal* if ρ is continuous for k isomorphic to $\mathbb{R}$; or if ρ is a *k-rational* map for k non-archimedean or isomorphic to $\mathbb{C}$.

There is a decomposition $G = G_c G_s$ (resp. $\tilde{G} = \tilde{G}_c \tilde{G}_s$) where G_c (resp. $\tilde{G}_c$) is the product of compact factors (resp. k-anisotropic factors) of G (resp. $\tilde{G}$) and G_s (resp. $\tilde{G}_s$) is the product of non-compact factors (resp. k-isotropic factors) of G (resp. $\tilde{G}$) when k is isomorphic to $\mathbb{R}$ (resp. non-archimedean or isomorphic to $\mathbb{C}$).

Denote by G_i (resp. $\tilde{G}_i$), $1 \leqslant i \leqslant j$, the non-compact factors (resp. k-isotropic factors) of G (resp. $\tilde{G}$). Also set $G_s = \tilde{G}_s(k)$ and $G_i = \tilde{G}_i(k)$ for k non-archimedean or isomorphic to $\mathbb{C}$. We call these G_i the *non-compact almost k-simple factors* of G.

Definition 1.1. A normal representation ρ of G on V is called *good* if the $\rho(G_s)$-fixed points in V are $\{0\}$; and ρ is called *excellent* if $\rho(G_i)$-fixed points in V are $\{0\}$ for each non-compact almost k-simple factor G_i of G.

In this paper we present an "upper bound function" for G-matrix coefficients for all unitary representations of $G \ltimes_\rho V$ without non-trivial V-fixed vectors if ρ is an excellent representation of G on V. Special cases of $SL(2,\mathbb{K}) \ltimes \mathbb{K}^2$ and $SL(2,\mathbb{K}) \ltimes \mathbb{K}^3$ are considered in [28] and [17] for a local field $\mathbb{K}$. For these cases the following conditions are satisfied: every orbit is locally closed (intersection of an open and a closed set) in the dual group $\widehat{V}$; and for each $\chi \in \widehat{V} \backslash \{0\}$ the stabilizer $S_\chi = \{g \in G \ltimes_\rho V\colon\ g \cdot \chi = \chi\}$ is amenable. The first one allows us to use the "Mackey machine" and the latter one implies that the G-matrix coefficients are bounded by the Harish-Chandra functions. Margulis also used this criterion in [26, Theorem 2] to prove Kazhdan's property (T) of the pair $(O_3(\mathbb{Q}_5) \ltimes \mathbb{Q}_3^5, \mathbb{Q}_3^5)$. In fact, if G is a connected almost k-algebraic group and ρ is a normal representation, then "Mackey machine" applies to the semidirect product $G \ltimes_\rho V$ and hence we have complete descriptions of the dual groups of $G \ltimes_\rho V$ (see [37, Theorem 7.3.1] or [35, Chapter 5.4]). Therefore any irreducible representation π of $G \ltimes_\rho V$ without non-trivial V-fixed vectors is induced from the ones on the stabilizers S_χ, $\chi \in \widehat{V} \backslash \{0\}$. However, for general cases, the complexity of these stabilizers S_χ would require heavy analysis calculations.

Our work is an extension of the ideas of R.E. Howe and E.C. Tan [17, Chapter V, Theorem 3.3.1]. For $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, they considered the system of imprimitivity based on $(SL(2,\mathbb{R}), \widehat{\mathbb{R}^2})$ instead of "Mackey machine" to calculate upper bounds of $SO(2)$-finite matrix coefficients restricted on $SL(2,\mathbb{R})$. More precisely, the deformation of $SO(2)$-orbits under the natural dual action of the Cartan subgroup on $\widehat{\mathbb{R}^2}$ gives