Journal of Functional Analysis 267 (2014) 80–111

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Local operator multipliers and positivity

N.M. Steen [a], I.G. Todorov [a,*], L. Turowska [b]

[a] *Pure Mathematics Research Centre, Queen's University Belfast, Belfast BT7 1NN, United Kingdom*
[b] *Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Sweden*

ARTICLE INFO

Article history:
Received 18 June 2012
Accepted 10 April 2014
Available online 5 May 2014
Communicated by D. Voiculescu

Keywords:
Positive
Local
Schur multiplier
Operator multiplier

ABSTRACT

We establish an unbounded version of Stinespring's Theorem and a lifting result for Stinespring representations of completely positive modular maps defined on the space of all compact operators. We apply these results to study positivity for Schur multipliers. We characterise positive local Schur multipliers, and provide a description of positive local Schur multipliers of Toeplitz type. We introduce local operator multipliers as a non-commutative analogue of local Schur multipliers, and characterise them extending both the characterisation of operator multipliers from [16] and that of local Schur multipliers from [27]. We provide a description of the positive local operator multipliers in terms of approximation by elements of canonical positive cones.

1. Introduction

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i,j)a_{i,j})$ is the matrix of a bounded linear operator on ℓ^2 whenever $(a_{i,j})$ is such. The study of Schur multipliers was initiated by I. Schur in the early 20th century, and a characterisation of

* Corresponding author.
E-mail addresses: nsteen01@qub.ac.uk (N.M. Steen), i.todorov@qub.ac.uk (I.G. Todorov), turowska@chalmers.se (L. Turowska).

http://dx.doi.org/10.1016/j.jfa.2014.04.007

these objects was given by A. Grothendieck in his *Résumé* [10] (see also [23]). A measurable version of Schur multipliers was developed by M.S. Birman and M.Z. Solomyak (see [3] and the references therein) and V.V. Peller [22]. More concretely, given standard measure spaces (X,μ) and (Y,ν) and a function $\varphi : X \times Y \to \mathbb{C}$, one defines a linear transformation S_φ on the space of all Hilbert–Schmidt operators from $H_1 = L^2(X,\mu)$ to $H_2 = L^2(Y,\nu)$ by multiplying their integral kernels by φ; if S_φ is bounded in the operator norm (in which case φ is called a *measurable Schur multiplier*), one extends it to the space $\mathcal{K}(H_1,H_2)$ of all compact operators from H_1 into H_2 by continuity. The map S_φ is defined on the space $\mathcal{B}(H_1,H_2)$ of all bounded linear operators from H_1 into H_2 by taking the second dual of the constructed map on $\mathcal{K}(H_1,H_2)$. A characterisation of measurable Schur multipliers, extending Grothendieck's result, was obtained in [22] (see also [18] and [29]). Namely, a function $\varphi \in L^\infty(X \times Y)$ was shown to be a Schur multiplier if and only if φ coincides almost everywhere with a function of the form $\sum_{k=1}^\infty a_k(x)b_k(y)$, where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are families of essentially bounded measurable functions such that $\operatorname{esssup}_{x\in X}\sum_{k=1}^\infty |a_k(x)|^2 < \infty$ and $\operatorname{esssup}_{y\in Y}\sum_{k=1}^\infty |b_k(y)|^2 < \infty$.

A local version of Schur multipliers was defined and studied in [27]. Local Schur multipliers are, in general, unbounded, but necessarily closable, densely defined linear transformations on $\mathcal{B}(L^2(X,\mu), L^2(Y,\nu))$. A measurable function $\varphi : X \times Y \to \mathbb{C}$ was shown in [27] to be a local Schur multiplier if and only if it agrees almost everywhere with a function of the form $\sum_{k=1}^\infty a_k(x)b_k(y)$, where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are families of measurable functions such that $\sum_{k=1}^\infty |a_k(x)|^2 < \infty$ for almost all $x \in X$ and $\sum_{k=1}^\infty |b_k(y)|^2 < \infty$ for almost all $y \in Y$.

In [19], a quantised version of Schur multipliers, called universal operator multipliers, was introduced. Universal operator multipliers are defined as elements of C*-algebras satisfying certain boundedness conditions, and hence are non-commutative versions of continuous Schur multipliers. A characterisation of universal operator multipliers, generalising Grothendieck–Peller's results, was obtained in [15].

In the present paper, we introduce and study local operator multipliers. Due to their spatial nature, the natural setting here is that of von Neumann algebras. Pursuing the analogue with the commutative setting, where local multipliers are measurable (not necessarily bounded) functions of two variables, we define local operator multipliers as operators affiliated with the tensor product of two von Neumann algebras. We characterise local operator multipliers, extending both the description of local Schur multipliers from [27] and the description of universal operator multipliers from [15]. We further characterise the positive local Schur multipliers (Section 4), as well as the positive local operator multipliers (Section 6). We describe positive local Schur multipliers of Toeplitz type, and consider local Schur multipliers that are divided differences, that is, functions of the form $\frac{f(x)-f(y)}{x-y}$. We show that such a divided difference is a positive local Schur multiplier with respect to every standard Borel measure if and only if f is an operator monotone function.

Our main tool for characterising positivity of multipliers is an unbounded version of Stinespring's Theorem (Section 2). In the literature, there are a number of versions of