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Global functional calculus for operators on compact Lie groups

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In this paper we develop the functional calculus for elliptic operators on compact Lie groups without the assumption that the operator is a classical pseudo-differential operator. Consequently, we provide a symbolic descriptions of complex powers of such operators. As an application, we give a constructive symbolic proof of the Gårding inequality for operators in (ρ, δ)-classes in the setting of compact Lie groups.

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1. Introduction

In [23], Seeley has developed the functional calculus of classical pseudo-differential operators on compact manifolds. The main idea of the construction was to define functions of homogeneous components of the symbol in local coordinates, and then patch them together obtaining a globally defined function of an operator.

Over the years, this idea has been developed further for elliptic operators in different settings, see e.g. Kumano-go and Tsutsumi [10], Beals [1], Kumano-go [9], Helffer [8], Coriasco, Schrohe and Seiler [4], to mention only very few contributions. On manifolds with particular geometries (of the manifold itself or of its boundary) the construction of complex powers of operators becomes adapted to the underlying geometry, see e.g. Schrohe [21,22] for geometries with fibred boundaries, or Loya [12,14] for manifolds with conical singularities, where in the latter the analysis is based on the heat kernel techniques [13]. There are important applications, such as those of the ζ-function of an operator T (defined by $\zeta(z) = \operatorname{trace} T^z$) to the index theory, or to the evolution equations. There are further applications of complex powers of operators to Wodzicki-type residues, see e.g. Buzano and Nicola [3] for a good and more extensive literature review of the above topics, as well as for the complex powers in the Weyl–Hörmander calculus.

In this paper we show that in the setting of compact Lie groups one can work with functions of operators using the globally defined matrix symbols instead of representations in local coordinates, which is the version of the analysis well adopted to the operator theory on compact Lie groups. These matrix symbols and their calculus have been recently developed in [16,18], and in [19] the characterisation of operators in Hörmander's classes $\Psi^m = \Psi^m_{1,0}$ on the compact Lie group viewed as a manifold was given in terms of these matrix symbols, thus providing a link between local and global symbolic calculi. In Section 2 we briefly review the required parts of these constructions. In particular, notions such as the ellipticity and hypoellipticity can be characterised in terms of the matrix symbols. The matrix symbols have been instrumental in handling other problems, for example for proving the Hörmander–Mikhlin multiplier theorem [20] in the setting of general compact Lie groups.

In order to approach the functional calculus of operators from the point of view of the symbolic calculus of matrix symbols, first we introduce a notion of parameter-dependent ellipticity in our setting and investigate its properties. Consequently, we apply it to defining the functions of matrix symbols which are then quantised to provide functions of operators. If the operators are sufficiently nice, for example self-adjoint, the obtained functions of operators coincide with those that can be defined by the spectral calculus.

Therefore, we can note that the proposed approach applies to a wide class of operators which, in particular, do not have to be self-adjoint, and do not have to be classical pseudo-differential operators. In fact, we can work with the families of (ρ, δ)-classes defined in terms of the matrix symbols. In the case of $\rho = 1$ and $\delta = 0$, this class coincides with the usual Hörmander class of pseudo-differential operators on compact