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On the spectrum of shear flows and uniform ergodic theorems

Jonathan Ben-Artzi

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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ABSTRACT

The spectra of parallel flows (that is, flows governed by first-order differential operators parallel to one direction) are investigated, on both L^2 spaces and weighted-L^2 spaces. As a consequence, an example of a flow admitting a purely singular continuous spectrum is provided. For flows admitting more regular spectra the density of states is analyzed, and spaces on which it is uniformly bounded are identified. As an application, an ergodic theorem with uniform convergence is proved.

1. Introduction

We study operators of the form

$$H_1 = -i\psi \frac{\partial}{\partial x_1} \quad \text{acting in } L^2(\mathbb{R}^d) \tag{1.1}$$

E-mail address: J.Ben-Artzi@damtp.cam.ac.uk.

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and

$$H_w = -i\frac{\psi}{w}\frac{\partial}{\partial x_1} \quad \text{acting in } L^2_w\big(\mathbb{R}^d\big) \tag{1.2}$$

where $0 < \psi \in L^\infty(\mathbb{R}^d)$ depends only on $x' = (x_2, \dots, x_d)$, $0 < w \in L^\infty(\mathbb{R}^d)$ is a weight function and L^2_w is the weighted-L^2 space endowed with the inner product

$$(f, g)_{L^2_w} = \int\limits_{\mathbb{R}^d} f\bar{g}w.$$

We naturally call such operators *shear flows* which is the standard term in fluid dynamics for flows that have straight and parallel flow lines. Since ψ and w are assumed to be real-valued both H_1 and H_w are symmetric. Self-adjointness of H_1 (under mild conditions on ψ) is standard (Corollary 2.6), but that is not the case for H_w. In Theorem 3.5 we give sufficient conditions on w and find an appropriate domain so that H_w is self-adjoint. We characterize the spectrum of H_w and give an explicit example where the spectrum is purely singular continuous.

In cases where the spectrum is more well-behaved, we study the density of states of both operators and identify spaces $\mathcal{X}^\sigma \subset L^2(\mathbb{R}^d)$ and $\mathcal{Y}^\sigma \subset L^2_w(\mathbb{R}^d)$ on which there is an explicit estimate for the density of states of both operators (the parameter $\sigma \in \mathbb{R}$ is related to the behavior at infinity). Letting $G_t = e^{itH_w}$ be the one-parameter unitary group of transformations generated by H_w and letting P_w be the orthogonal projection onto $\{f \in L^2_w(\mathbb{R}^d) \mid f \circ G_t = f\}$, we use the estimate on the density of states to obtain the *uniform* convergence

$$\lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} G_t\, dt = P_w \quad \text{in } \mathcal{B}\big(\mathcal{Y}^\sigma, \mathcal{Y}^{-\sigma}\big)$$

where $\mathcal{Y}^{-\sigma}$ is the dual space to $\mathcal{Y}^\sigma$ with respect to the inner product in L^2_w and $\mathcal{B}(\mathcal{Y}^\sigma, \mathcal{Y}^{-\sigma})$ denotes the space of bounded linear operators from $\mathcal{Y}^\sigma$ to $\mathcal{Y}^{-\sigma}$. The proof follows the ideas of von Neumann [15] in his proof of the ergodic theorem. It diverges from von Neumann's proof in that we replace the Stieltjes measure $d(E(\lambda)f, g)$ by its density using the estimates on the density of states (where $E(\lambda)$, $\lambda \in \mathbb{R}$, is the spectral family).

The existing literature on spectra of first-order differential operators, it appears, has primarily been in relation to Euler's equations for incompressible fluids, in particular in two dimensions and in bounded domains. Recently, Cox [3] studied the spectrum of the linearization of Euler's equations in the vorticity formulation. Specifically, he focused on the spectrum due to *periodic* trajectories of the flow. These results are analogous to our results for *unbounded* trajectories in *weighted* spaces, see Corollary 3.6. We refer to the references within [3] as well as the survey article [13] for further discussion.