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Quasiaffine orbits of invariant subspaces for uniform Jordan operators

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ABSTRACT

We consider the problem of classification of invariant subspaces for the class of uniform Jordan operators. We show that given two invariant subspaces M_1 and M_2 of a uniform Jordan operator $T = S(\theta) \oplus S(\theta) \oplus \cdots$, the subspace M_2 belongs to the quasiaffine orbit of M_1 if and only if the restrictions $T|M_1$ and $T|M_2$ are quasisimilar and the compression $T_{M_2^\perp}$ can be injected in the compression $T_{M_1^\perp}$. Our result refines previous work on the subject by Bercovici and Smotzer.

1. Introduction

Let $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be bounded linear operators on Hilbert spaces. If M_1 and M_2 are invariant subspaces for T_1 and T_2 respectively (that is $M_1 \subset \mathcal{H}_1$ and $M_2 \subset \mathcal{H}_2$ are closed subspaces such that $T_1 M_1 \subset M_1$ and $T_2 M_2 \subset M_2$), we say that M_1 is a *quasiaffine transform* of M_2 if there exists a bounded injective operator with dense range $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2 X$ and $\overline{XM_1} = M_2$. We write $M_1 \prec M_2$ when M_1 is a quasiaffine transform of M_2. In that case, we also say that M_2 lies in the *quasiaffine orbit* of M_1. When $M_1 \prec M_2$ and $M_2 \prec M_1$, we say that M_1 and M_2 are *quasisimilar* and write $M_1 \sim M_2$. Quasisimilarity is clearly an equivalence relation on the class of pairs of the form (T, M), where M is an invariant subspace for the bounded

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linear operator T. In [2] (see Problem 5.2), Bercovici raised the basic question underlying our present investigation: describe the quasiaffine orbit of a given invariant subspace for an operator of class C_0 (the definition of which will be recalled in Section 2).

Related results for general operators of class C_0 can be found in [2], where it is proved that the quasisimilarity class of an invariant subspace is determined by the quasisimilarity class of the restriction $T|M$ if and only if T has a certain finiteness property (namely property (Q) introduced in [12]). Nilpotent operators of finite multiplicity have been considered in [10]. In that context, it was proved that the quasisimilarity class of M is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression $T_{M^\perp}$ when either of those operators has multiplicity one. In addition, the authors of [10] considered a combinatorial object (a sequence of partitions) known as a *Littlewood–Richardson sequence* which encodes the relationships that must hold between the Jordan models of T, $T|M$ and $T_{M^\perp}$ (see also [7–9]). Using these objects, they prove that for multiplicity at least three, the quasisimilarity classes of $T|M$ and $T_{M^\perp}$ are not sufficient to determine the quasisimilarity class of M. From a slightly different point of view, it was proved in [7] that if M_1 and M_2 are cyclic invariant subspaces for T, then they must be quasiaffine transforms of a common cyclic invariant subspace N (in other words, M_1 and M_2 lie in the same weakly quasiaffine orbit) whenever the restrictions (respectively, the compressions) of T to M_1 and M_2 are quasisimilar.

The objects we will be concerned with in this work are the so-called uniform Jordan operators: that is operators of the form

$$T = S(\theta) \oplus S(\theta) \oplus \cdots.$$

These operators are interesting since any C_0 contraction is the compression of a uniform Jordan operator, where θ is the minimal function of T. This well-known fact follows easily from considerations related to the minimal isometric dilation of T, and we refer the reader to [11] for greater detail. In addition, uniform Jordan operators appear to be more amenable and our understanding of the quasisimilarity classes of their invariant subspaces is significantly better, thanks to the pioneer work of Bercovici and Tannenbaum (see [4]). Indeed, motivated by interpolation problems from [5] and [6], they considered the case where the Jordan operator T has finite multiplicity and established that $M_1 \sim M_2$ if and only if $T|M_1 \sim T|M_2$. Moreover, it was observed that for the operator $T = S(z^2) \oplus S(z)$ this classification breaks down, so the corresponding result fails if T is not uniform. Later on, it was proved in [2] that this classification holds for a uniform Jordan operator T if and only if $T|M$ satisfies property (P), another finiteness property which is stronger than the aforementioned property (Q). In general, the quasisimilarity class of an invariant subspace for a uniform Jordan operator is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression $T_{M^\perp}$ (see [3]).

We focus in this paper on the weaker notion of quasiaffine orbit. After presenting the necessary preliminaries in Section 2, we prove in Section 3 our main theorem (Theorem 3.4) which gives a characterization of these orbits for uniform Jordan operators