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On the quasinodal map for the diffusion operator

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ABSTRACT

We study inverse nodal problems for the diffusion operator with separated boundary conditions. It is shown that the space of diffusion operators characterized by $B := B(p,q,\alpha,\beta)$, where $(p,q,\alpha,\beta) \in W_1^1(0,\pi) \times L^1(0,\pi) \times [0,\pi)^2$, under a certain metric, is homeomorphic (i.e., continuous, open bijection between two topological spaces) to the partition set of the space of quasinodal sequences. As a consequence, the inverse nodal problem defined on the partition set of admissible sequence, is stable.

1. Introduction

The inverse nodal problem was posed and solved for Sturm–Liouville problems by McLaughlin [21], who showed that knowledge of a dense subset of nodal points (zeros) of the eigenfunctions alone can determine the potential function of the Sturm–Liouville problem up to a constant. This is the so-called inverse nodal problem [15]. Recently, some authors have reconstructed the potential function for generalizations of the Sturm–Liouville problem from the nodal points (refer to [2,4–11,15,17–21,25,24,26–28,30–33,35]).

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The problem of describing the interactions between colliding particles is of fundamental interest in physics. One is interested in collisions of two spinless particles, and it is supposed that the s-wave scattering matrix and the s-wave binding energies are exactly known from collision experiments. With a radial static potential $V(x)$ the s-wave Schrödinger equation is written as

$$y''(x) + \big[E - V(E,x)\big]y(x) = 0,$$

where $V(E,x)$ has the following form of the energy dependence

$$V(E,x) = 2\sqrt{E}P(x) + Q(x).$$

In this paper we consider the diffusion operator $B := B(p,q,\alpha,\beta)$ written as

$$-y''(x) + \big[q(x) + 2\lambda p(x)\big]y(x) = \lambda^2 y(x), \quad x \in [0,\pi],$$

where real-valued functions $p \in W_1^1[0,\pi]$ and $q \in L^1[0,\pi]$, and with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \alpha,\beta \in [0,\pi).$$

For simplicity, throughout this paper we may assume that $\int_0^\pi p(x)\,dx = 0$ mathematically. This normalization can be realized by translation. It is well known that for sufficiently large $|n|$ the spectrum $\{\lambda_n\}$ of the problem B is real and simple [16,28,29]. Let the $y_n(x)$ be the eigenfunction of the problem B corresponding to the eigenvalue λ_n. We note that for sufficiently large n the eigenfunction $y_n(x)$ is real-valued function and the eigenfunction $y_n(x)$ has exactly $n-1$ (simple) zeros inside the interval $(0,\pi)$, namely: $0 < x_n^1 < \cdots < x_n^{n-1} < \pi$. The set $X := \{x_n^j\}_{n\geqslant 2,\, j=\overline{1,n-1}}$ is called the set of nodal points of the boundary value problem B.

Oscillation theory for a quadratic eigenvalue problem was solved by Browne and Watson in [3]. Gesztesy and Holden derived an infinite sequence of trace formulas associated with it, employing methods familiar from supersymmetric quantum mechanics in [13]. Eigenvalue asymptotics, identities for eigenvalues of the diffusion operator were established in [1,22,29]. Yurko established properties of the spectral characteristics and investigated the inverse spectral problem of recovering the coefficients of differential pencils on compact graphs from the so-called Weyl vector which is a generalization of the Weyl function (m-function) for the classical Sturm–Liouville operator in [34]. Determination of the diffusion operator from spectral data was given in [4,5,12,14,16,23,36].

Under the condition that $p(x)$ is known a prior [17] one obtained a uniqueness theorem on the boundary condition parameters and the potential function $q(x)$ of the diffusion operator by nodal data. After that, the obvious question occurs: what if boundary condition parameters and $p(x)$, $q(x)$ are all unknown? Our motivation in considering nodal points of eigenfunctions as data is our desire to obtain “more” information on the diffusion operator. In [28], we proved that a dense subset of nodal data can uniquely determine the