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Lie ring isomorphisms between nest algebras on Banach spaces ☆

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ABSTRACT

Let $\mathcal{N}$ and $\mathcal{M}$ be nests on Banach spaces X and Y over the (real or complex) field $\mathbb{F}$ and let $\operatorname{Alg}\mathcal{N}$ and $\operatorname{Alg}\mathcal{M}$ be the associated nest algebras, respectively. It is shown that a map $\Phi : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{M}$ is a Lie ring isomorphism (i.e., Φ is additive, Lie multiplicative and bijective) if and only if Φ has the form $\Phi(A) = TAT^{-1} + h(A)I$ for all $A \in \operatorname{Alg}\mathcal{N}$ or $\Phi(A) = -TA^*T^{-1} + h(A)I$ for all $A \in \operatorname{Alg}\mathcal{N}$, where h is an additive functional vanishing on all commutators and T is an invertible bounded linear or conjugate linear operator when $\dim X = \infty$; T is a bijective τ-linear transformation for some field automorphism τ of $\mathbb{F}$ when $\dim X < \infty$.

1. Introduction and main results

Let $\mathcal{R}$ and $\mathcal{R}'$ be two associative rings. Recall that a map $\phi : \mathcal{R} \to \mathcal{R}'$ is called a multiplicative map if $\phi(AB) = \phi(A)\phi(B)$ for any $A, B \in \mathcal{R}$; is called a Lie multiplicative map if $\phi([A, B]) = [\phi(A), \phi(B)]$ for any $A, B \in \mathcal{R}$, where $[A, B] = AB - BA$ is the Lie product of A and B which is also called a commutator. In addition, a map $\phi : \mathcal{R} \to \mathcal{R}'$ is called a Lie multiplicative isomorphism if ϕ is bijective and Lie multiplicative; is called

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a Lie ring isomorphism if ϕ is bijective, additive and Lie multiplicative. If $\mathcal{R}$ and $\mathcal{R}'$ are algebras over a field $\mathbb{F}$, $\phi : \mathcal{R} \to \mathcal{R}'$ is called a Lie algebraic isomorphism if ϕ is bijective, $\mathbb{F}$-linear and Lie multiplicative. For the study of Lie ring isomorphisms between rings, see [3,5,10] and the references therein. In this paper we focus our attention on Lie ring isomorphisms between nest algebras on general Banach spaces.

Let X be a Banach space over the (real or complex) field $\mathbb{F}$ with topological dual X^*. $\mathcal{B}(X)$ stands for the algebra of all bounded linear operators on X. A nest $\mathcal{N}$ on X is a complete totally ordered subspace lattice, that is, a chain of closed (under norm topology) subspaces of X which is closed under the formation of arbitrary closed linear span (denote by $\bigvee$) and intersection (denote by $\bigwedge$), and which includes (0) and X. The nest algebra associated with a nest $\mathcal{N}$, denoted by $\operatorname{Alg}\mathcal{N}$, is the weakly closed operator algebra consisting of all operators that leave every subspace $N \in \mathcal{N}$ invariant. For $N \in \mathcal{N}$, let $N_- = \bigvee\{M \in \mathcal{N} \mid M \subset N\}$ and $N_-^{\perp} = (N_-)^{\perp}$, where $N^{\perp} = \{f \in X^* \mid N \subseteq \ker(f)\}$. If $\mathcal{N}$ is a nest on X, then $\mathcal{N}^{\perp} = \{N^{\perp} \mid N \in \mathcal{N}\}$ is a nest on X^* and $(\operatorname{Alg}\mathcal{N})^* \subseteq \operatorname{Alg}\mathcal{N}^{\perp}$. If $\mathcal{N} = \{(0), X\}$, we say that $\mathcal{N}$ is a trivial nest, in this case, $\operatorname{Alg}\mathcal{N} = \mathcal{B}(X)$. Non-trivial nest algebras are very important reflexive operator algebras that are not semi-simple, not semi-prime and not self-adjoint. If $\dim X < \infty$, a nest algebra on X is isomorphic to an algebra of upper triangular block matrices. Nest algebras are studied intensively by a lot of literatures. For more details on basic theory of nest algebras, the readers can refer to [6,8].

In [9], Marcoux and Sourour proved that every Lie algebraic isomorphism between nest algebras on separable complex Hilbert spaces is a sum $\alpha + \beta$, where α is an algebraic isomorphism or the negative of an algebraic anti-isomorphism and $\beta : \operatorname{Alg}\mathcal{N} \to \mathbb{C}I$ is a linear map vanishing on all commutators, that is, satisfying $\beta([A,B]) = 0$ for all $A, B \in \operatorname{Alg}\mathcal{N}$.

Qi and Hou in [11] generalized the result of Marcoux and Sourour by classifying certain Lie multiplicative isomorphisms. Note that, a Lie multiplicative isomorphism needs not be additive. Let $\mathcal{N}$ and $\mathcal{M}$ be nests on Banach spaces X and Y over the (real or complex) field $\mathbb{F}$, respectively, with the property that if $M \in \mathcal{M}$ such that $M_- = M$, then M is complemented in Y (obviously, this assumption is not needed if Y is a Hilbert space or if $\dim Y < \infty$). Let $\operatorname{Alg}\mathcal{N}$ and $\operatorname{Alg}\mathcal{M}$ be respectively the associated nest algebras, and let $\Phi : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{M}$ be a bijective map. Qi and Hou in [11] proved that, if $\dim X = \infty$ and if there is a nontrivial element in $\mathcal{N}$ which is complemented in X, then Φ is a Lie multiplicative isomorphism if and only if there exists a map $h : \operatorname{Alg}\mathcal{N} \to \mathbb{F}I$ with $h([A,B]) = 0$ for all $A, B \in \operatorname{Alg}\mathcal{N}$ such that Φ has the form $\Phi(A) = TAT^{-1} + h(A)$ for all $A \in \operatorname{Alg}\mathcal{N}$ or $\Phi(A) = -TA^*T^{-1} + h(A)$ for all $A \in \operatorname{Alg}\mathcal{N}$, where, in the first form, $T : X \to Y$ is an invertible bounded linear or conjugate-linear operator so that $N \mapsto T(N)$ is an order isomorphism from $\mathcal{N}$ onto $\mathcal{M}$, while in the second form, X and Y are reflexive, $T : X^* \to Y$ is an invertible bounded linear or conjugate-linear operator so that $N^{\perp} \mapsto T(N^{\perp})$ is an order isomorphism from $\mathcal{N}^{\perp}$ onto $\mathcal{M}$. If $\dim X = n < \infty$, identifying nest algebras with upper triangular block matrix algebras, then Φ is a Lie multiplicative isomorphism if and only if there exist a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and