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On the completion problem for algebra H^∞ ☆

Alexander Brudnyi

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, T2N 1N4, Canada

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ABSTRACT

We study the completion problem to an invertible operator-valued function for the class of bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{S}$ with relatively compact images in the space of bounded linear operators between complex Banach spaces. In particular, we prove that in this class of functions the operator-valued corona problem and the completion problem are not equivalent, and establish an Oka-type principle asserting that the completion problem is solvable if and only if it is solvable in the class of continuous operator-valued functions on $\mathbb{D}$ with relatively compact images.

1. Introduction

1.1. Corona problem

Let H^∞ be the Banach space of bounded holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ equipped with the supremum norm and $H^\infty(L(X,Y))$ be the Banach space of holomorphic functions F on $\mathbb{D}$ with values in the space of bounded linear operators $X \to Y$ between complex Banach spaces X, Y with norm $\|F\| := \sup_{z\in\mathbb{D}} \|F(z)\|_{L(X,Y)}$. By I_X we denote the identity operator $X \to X$.

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E-mail address: albru@math.ucalgary.ca.

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The following problem was posed by Sz.-Nagy [20] in 1978:

Operator Corona Problem. Suppose $F \in H^\infty(L(H_1, H_2))$, where H_1, H_2 are separable Hilbert spaces, satisfies $\|F(z)x\| \geqslant \delta\|x\|$ for all $x \in H_1$, $z \in \mathbb{D}$, where $\delta > 0$ is a constant. Does there exist $G \in H^\infty(L(H_2, H_1))$ such that $G(z)F(z) = I_{H_1}$ for all $z \in \mathbb{D}$?

This problem is of importance in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions) and in control theory (the stabilization problem). It is also related to the study of submodules of H^∞ and to many other subjects of analysis, see [16,15,21,22,27] and references therein. Obviously, the condition imposed on F is necessary since it implies existence of a uniformly bounded family of left inverses of $F(z)$, $z \in \mathbb{D}$. The question is whether this condition is sufficient for the existence of a bounded holomorphic left inverse of F. In general, the answer is known to be negative (see [21,23,25] and references therein). But in some specific cases it is positive. In particular, the famous Carleson corona theorem [5] stating that for $\{f_i\}_{i=1}^n \subset H^\infty$ the Bezout equation $\sum_{i=1}^n g_i f_i = 1$ is solvable with $\{g_i\}_{i=1}^n \subset H^\infty$ as soon as $\max_{1\leqslant i\leqslant n} |f_i(z)| > \delta > 0$ for every $z \in \mathbb{D}$ means that the answer is positive when $\dim H_1 = 1$, $\dim H_2 = n < \infty$. More generally, the answer is positive as soon as $\dim H_1 < \infty$ or F is a "small" perturbation of a left invertible function $F_0 \in H^\infty(L(H_1, H_2))$ (for example, if $F - F_0$ belongs to $H^\infty(L(H_1, H_2))$ with values in the class of Hilbert Schmidt operators), see [24] and references therein.

For a long time there were no positive results in the case $\dim H_1 = \infty$. The first positive results were obtained in [28] where the following more general problem was studied.

Problem 1. Let X_1, X_2 be complex Banach spaces and $F \in H^\infty(L(X_1, X_2))$ be such that for each $z \in \mathbb{D}$ there exists a left inverse G_z of $F(z)$ satisfying $\sup_{z\in\mathbb{D}} \|G_z\| < \infty$. Does there exist $G \in H^\infty(L(X_2, X_1))$ such that $G(z)F(z) = I_{X_1}$ for all $z \in \mathbb{D}$?

Since in this general setting the answer is negative, it was suggested in [28] to investigate the problem for the case of $F \in H^\infty_{\mathrm{comp}}(L(X_1, X_2))$, the space of holomorphic functions on $\mathbb{D}$ with relatively compact images in $L(X_1, X_2)$.[1] In particular, it was shown, see [28, Th. 2.1], that the answer is positive for F that can be uniformly approximated by finite sums $\sum f_k(z)L_k$, where $f_k \in H^\infty$ and $L_k \in L(X_1, X_2)$. The question of whether each $F \in H^\infty_{\mathrm{comp}}(L(X_1, X_2))$ can be obtained in that form is closely related to the still open problem about the Grothendieck approximation property for H^∞. (The strongest result in this direction [2, Th. 9] states that H^∞ has the approximation property "up to logarithm".) Another, not involving the approximation property for H^∞, approach that led to the solution of Problem 1 for H^∞_{comp} spaces was proposed in [4].

[1] The notation H^∞_{comp} had been introduced in [28].