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Local energy decay for the damped wave equation

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A B S T R A C T

We prove local energy decay for the damped wave equation on $\mathbb{R}^d$. The problem which we consider is given by a long range metric perturbation of the Euclidean Laplacian with a short range absorption index. Under a geometric control assumption on the dissipation we obtain an almost optimal polynomial decay for the energy in suitable weighted spaces. The proof relies on uniform estimates for the corresponding "resolvent", both for low and high frequencies. These estimates are given by an improved dissipative version of Mourre's commutators method.

Contents

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1. Introduction

We consider on $\mathbb{R}^d$, $d \geqslant 3$, the damped wave equation:

$$\begin{cases} \partial_t^2 u(t,x) + H_0 u(t,x) + a(x)\partial_t u(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) & \text{for } x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here H_0 is an operator in divergence form

$$H_0 = -\operatorname{div}\bigl(G(x)\nabla\bigr),$$

where $G(x)$ is a positive symmetric matrix with smooth entries, which is a long range perturbation of the identity (see (1.2)). Laplace–Beltrami operators will be considered as well, but the case of operators in divergence form captures all the difficulties. The operator H_0 is self-adjoint and non-negative on $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. The function $a \in C^\infty(\mathbb{R}^d)$ is the absorption index. It takes non-negative values and is a short range potential. More precisely we assume that there exists $\rho > 0$ such that for $j,k \in [\![1,d]\!]$, $\alpha \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ we have

$$\bigl|\partial^\alpha\bigl(G_{j,k}(x) - \delta_{j,k}\bigr)\bigr| \leqslant c_\alpha \langle x\rangle^{-\rho-|\alpha|} \quad \text{and} \quad \bigl|\partial^\alpha a(x)\bigr| \leqslant c_\alpha \langle x\rangle^{-1-\rho-|\alpha|}, \tag{1.2}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, $\delta_{j,k}$ is the Kronecker delta and $\mathbb{N}$ is the set of non-negative integers.

Let $\mathcal{H}$ be the Hilbert completion of $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ for the norm

$$\bigl\|(u,v)\bigr\|_{\mathcal{H}}^2 = \bigl\|H_0^{1/2}u\bigr\|_{L^2}^2 + \|v\|_{L^2}^2. \tag{1.3}$$

Here we use the square root $H_0^{1/2}$ of the self-adjoint operator H_0 but the corresponding term in the above energy can also be written $\langle G(x)\nabla u, \nabla u\rangle_{L^2}$. Then $\mathcal{H} = \dot H^1 \times L^2$, $\dot H^1$ being the usual homogeneous Sobolev space on $\mathbb{R}^d$. We consider on $\mathcal{H}$ the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ H_0 & -ia \end{pmatrix} \tag{1.4}$$

with domain