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Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory

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A B S T R A C T

We present a study of what may be called an intrinsic metric for a general regular Dirichlet form. For such forms we then prove a Rademacher type theorem. For strongly local forms we show existence of a maximal intrinsic metric (under a weak continuity condition) and for Dirichlet forms with an absolutely continuous jump kernel we characterize intrinsic metrics by bounds on certain integrals. We then turn to applications on spectral theory and provide for (measure perturbation of) general regular Dirichlet forms an Allegretto–Piepenbrink type theorem, which is based on a ground state transform, and a Shnol' type theorem. Our setting includes Laplacian on manifolds, on graphs and α-stable processes.

1. Introduction

Intrinsic metrics play an important role in the study of various features of Laplacians on manifolds and more generally of Laplacians arising from strongly local Dirichlet forms.

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In this context, they appear for example in the study of heat kernel estimates [36,37], the investigation of stochastic completeness, recurrence and transience [35] and spectral theory, see e.g. [6,5,26]. Thus, basic issues for intrinsic metrics can be considered to be well understood in the case of strongly local Dirichlet forms. For non-local Dirichlet forms the situation is completely different. In fact, already for the simplest examples, viz. graphs, there is no common concept of an intrinsic metric (see, however, [8] for various ideas in this direction). This is the starting point for this paper.

Our basic aim is to propose an extension of the concept of intrinsic metric from strongly local Dirichlet forms to the general case and to study some of its basic features. More precisely, we proceed as follows:

After presenting the basic ingredients of Dirichlet forms in Section 2, we carry out a careful study of energy measure and of a suitable space of functions to be thought to belong locally to the domain in Section 3. In particular, we prove a certain continuity of the energy measures in Proposition 2.2 and discuss variants of the Leibniz rule.

In Section 4 we then present a general concept of intrinsic metric and study some of its properties. In particular, in Theorem 4.9 we provide a Rademacher type theorem in a rather general context. This theorem has already proven useful in Stollmann's study of length spaces [34]. In Section 5, we show that specifying an intrinsic metric more or less amounts to specifying a set of Lipschitz continuous functions.

Combining these results we then have a look at the strongly local case in our context in Section 6. In this case it is possible to show existence of a maximal intrinsic metric (Theorem 6.1). Existence of a maximal intrinsic metric fails in general for the non-local case, as we show by examples. In this sense, our results 'prove' that the non-local case is strictly more complicated than the local case as far as intrinsic metrics are concerned.

The situation of an absolutely continuous jump kernel is considered in Section 7. There, we can then characterize our intrinsic metrics by some integral type condition (Theorem 7.3). This is well in line with earlier results and ideas on e.g. graphs.

Dirichlet forms with finite jump size are considered in Section 8. In a precise sense, these turn out to be not much different from strongly local forms.

After these more geometric considerations we turn to spectral theory and present two applications of the developed theory. Both applications rely on the notion of generalized eigenfunction which in turn is defined using the local domain of definition. In their context, we actually allow furthermore for some perturbation of the original Dirichlet form by a (suitable) measure.

The first application, given in Section 10 provides a ground state transform and then an Allegretto–Piepenbrink type result. This basically unifies the corresponding results of [14,26] for graphs and strongly local forms respectively.

The second application concerns a Shnol' type result. Such a result was recently shown in [6] for strongly local forms using cut-off functions induced by the intrinsic metric. Having the intrinsic metrics at our disposal, we can adapt the strategy of [6] to our general context.