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Ginzburg–Landau functionals and renormalized energy: A revised Γ-convergence approach

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ABSTRACT

We give short and self-contained proofs of Γ-convergence results for Ginzburg–Landau energy functionals in two dimensions, in the logarithmic energetic regime. In particular, we derive the renormalized energy by Γ-convergence.

1. Introduction

The analysis of Ginzburg–Landau functionals as the length-scale parameter tends to zero is a beautiful piece of mathematical analysis. Different ideas and techniques merged together along the last twenty years, to give a clear picture of the relevant phenomena, as concentration of energy and formation of topological singularities.

To make a long story very short, the analysis started with the study of the asymptotic behavior of the minimizers of Ginzburg–Landau functionals in two dimensions, with prescribed boundary conditions. Let $\Omega \subset \mathbb{R}^2$ be bounded and with smooth boundary. The Ginzburg–Landau functionals $GL_\varepsilon : H^1(\Omega;\mathbb{R}^2) \to [0,+\infty]$, are defined as

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$$GL_\varepsilon(u) := \left(\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2} W\bigl(|u|\bigr) \right), \tag{1.1}$$

where $W \in C^0[0, +\infty)$ is such that $W(t) \geqslant 0$, $W^{-1}\{0\} = \{1\}$ and

$$\liminf_{t\to 1} \frac{W(t)}{(1-t)^2} > 0, \qquad \liminf_{t\to\infty} W(t) > 0.$$

The monograph [2] collects the main results about the asymptotic behavior of the minimizers u_ε of GL_ε with a prescribed boundary datum $g : \partial\Omega \mapsto S^1$ with degree d. For ε small, the energy $GL_\varepsilon(u_\varepsilon)$ blows up as $d|\log\varepsilon|$, and d vortex-like singularities appear, around which u_ε looks like (a fixed rotation of) $x/|x|$. Subtracting the leading term $d|\log\varepsilon|$ from the energy, a finite quantity remains in the limit, called *renormalized energy*, depending on the position of the singularities.

After these results, much work has been devoted to understand the behavior of sequences of non-minimizers, with prescribed energetic regime, in the spirit of Γ-convergence. The main issues are clearly the (zero order) Γ-convergence of $\frac{GL_\varepsilon}{|\log\varepsilon|}$ and the (first order) Γ-convergence of $GL_\varepsilon - d|\log\varepsilon|$ to the renormalized energy. The picture is nowadays well understood. For the zero order Γ-convergence [10] and [7] provide sharp lower bounds, while in [7] and [8] a Γ-convergence result is proved in $W^{1,1}(\Omega; R^2)$, and compactness of the singularities is expressed in terms of compactness properties of the Jacobians Ju_ε of u_ε in the dual of Hölder functions; in [1] the Γ-convergence result is obtained (in any dimension and codimension) with respect to the flat convergence of the Jacobians. On the other hand, lower bounds of the Ginzburg–Landau energy in terms of the renormalized energy have played an important role for the analysis of dynamics of vortices [6,9,13,14]. To our knowledge a self-contained proof of the first order Γ-convergence result is still missing.

The aim of this paper is to revisit these Γ-convergence results, giving short, efficient and self-contained proofs. Self-contained has to be understood in a very weak sense: we use many ideas from [1,10,7,8,11] and (for the first order Γ-limit) the analysis for minimizers developed in [2]. Our approach is the following: we consider the *ball construction* as done in [10]: it consists in an efficient way of selecting balls where the energy concentrates. Then, as in [11], we plug a Dirac mass in each ball, obtaining a sequence of measures μ_ε that approximates Ju_ε, carrying all the topological information and bringing compactness and Γ-liminf inequality in the zero order Γ-convergence result. The Γ-limsup inequality is obtained by a standard construction. To prove the first order Γ-convergence result we show that, if $GL_\varepsilon(u_\varepsilon)/|\log\varepsilon|$ is bounded, then around each singularity x_i we have $GL_\varepsilon(u_\varepsilon) - \pi z_i|\log\varepsilon| \geqslant C$, where $z_i \in \mathbb{Z}$ is the degree of the singularity. Moreover, if u_ε are optimal in energy, then $|z_i| = 1$ and around each singularity u_ε looks like (a rotation of) $\frac{x}{|x|}$. These preliminary results allow to use the analysis done in [2] to derive the renormalized energy.

Let us conclude recalling that the case treated in this paper has been the building block for a series of important generalizations, as for the case of external magnetic field [12],