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Quantitative uniqueness estimates for the general second order elliptic equations

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ABSTRACT

In this paper we study quantitative uniqueness estimates of solutions to general second order elliptic equations with magnetic and electric potentials. We derive lower bounds of decay rate at infinity for any nontrivial solution under some general assumptions. The lower bounds depend on asymptotic behaviors of magnetic and electric potentials. The proof is carried out by the Carleman method and bootstrapping arguments.

1. Introduction

In this paper we study the asymptotic behaviors of solutions to the general second order elliptic equation

$$Pv + W(x) \cdot \nabla v + V(x)v + q(x)v = 0 \quad \text{in } \Omega := \mathbb{R}^n \backslash \bar{B}, \tag{1.1}$$

where B is a bounded set in Ω. Here $P(x, D) = \sum_{jk} a_{jk}(x)\partial_j\partial_k$ is uniformly elliptic, i.e., for some $\lambda_0 > 0$

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$$\lambda_0|\xi|^2 \leqslant \sum_{jk} a_{jk}(x)\xi_j\xi_k \leqslant \lambda_0^{-1}|\xi|^2 \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{1.2}$$

and $a_{jk}(x)$ is Lipschitz continuous. We are interested in deriving lower bounds of the decay rate for any nontrivial solution v to (1.1) under certain a priori assumptions. This kind of problem was originally posed by Landis in the 60's [9]. He conjectured that if v is a bounded solution of

$$\Delta v + q(x)v = 0 \quad \text{in } \mathbb{R}^n \tag{1.3}$$

with $\|q\|_{L^\infty} \leqslant 1$ and $|v(x)| \leqslant C\exp(-C|x|^{1+})$ for some constant C, then v is identically zero. This conjecture was disproved by Meshkov [12] who constructed a $q(x)$ and a nontrivial $v(x)$ with $|v(x)| \leqslant C\exp(-C|x|^{4/3})$ satisfying (1.3). He also proved that if $|v(x)| \leqslant C_k\exp(-k|x|^{4/3})$ for all $k > 0$ then $v \equiv 0$. Note that $q(x)$ and $v(x)$ constructed by Meshkov are complex-valued. In 2005, Bourgain and Kenig [2] derived a quantitative version of Meshkov's result in their resolution of Anderson localization for the Bernoulli model. Precisely, they showed that if v is a bounded solution of $\Delta v + q(x)v = 0$ in $\mathbb{R}^n$ satisfying $\|q\|_{L^\infty} \leqslant 1$ and $v(0) = 1$, then

$$\inf_{|x_0|=R} \sup_{B(x_0,1)} \bigl|v(x)\bigr| \geqslant C\exp\bigl(-R^{4/3}\log R\bigr).$$

In view of Meshkov's example, the exponent $4/3$ is optimal.

Recently, Davey [4] derived similar quantitative asymptotic estimates for (1.1) with $P = -\Delta$ and $q(x) = -E \in \mathbb{C}$, i.e.,

$$-\Delta v + W(x)\cdot\nabla v + V(x)v = Ev. \tag{1.4}$$

To describe her result, we define

$$I(x_0) = \int_{|y-x_0|<1} \bigl|v(y)\bigr|^2\,dy$$

and

$$M(t) = \inf_{|x_0|=t} I(x_0).$$

Assume that $|V(x)| \lesssim \langle x\rangle^{-N}$ and $|W(x)| \lesssim \langle x\rangle^{-\tilde{p}}$, where $\langle x\rangle = \sqrt{1+|x|^2}$. Then it was shown that for any nontrivial bounded solution v of (1.4) with $v(0) = 1$, we have

$$M(t) \gtrsim \exp\bigl(-Ct^{\beta_0}(\log t)^{b(t)}\bigr), \tag{1.5}$$

where