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A lower bound criterion for iterated commutators

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ABSTRACT

We consider iterated commutators of multiplication by a symbol function and smooth Calderón–Zygmund operators, described by Fourier multipliers of homogeneity 0. We establish a criterion for a collection of symbols so that the corresponding Calderón–Zygmund operators characterize product BMO by means of iterated commutators. We therefore extend, in part, the line of one-parameter results following the work of Uchiyama and Li as well as the result in several parameters, concerning commutators with Riesz transforms by Lacey, Petermichl, Pipher, Wick.

1. Introduction

A classical result of Nehari [14] shows that a Hankel operator with antianalytic symbol b is bounded if and only if the symbol belongs to BMO. This theorem has an equivalent formulation by means of commutators of a symbol function b and the Hilbert transform, as the latter is a combination of orthogonal Hankel operators. Nehari's result leans on analytic structure in several crucial ways: the classical factorization result for

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H^1 functions on the disk and the fact that the Hilbert transform is a Fourier projection operator.

The classical text of Coifman, Rochberg and Weiss [5] extended the one-parameter theory to real analysis in the sense that the Hilbert transforms were replaced by Riesz transforms. In their text, they obtained sufficiency, i.e. that a BMO symbol b yields an L^2 bounded commutator for certain more general, convolution type singular integral operators. For necessity, they showed that the collection of Riesz transforms was representative enough. This is quite natural, in the view of the definition of H^1 requiring Riesz transforms being back in L^1 as well as the Fefferman–Stein decomposition of BMO using Riesz kernels.

Uchiyama [17] revisited said decomposition, with a very technical but constructive proof. It remarkably replaced the class of Riesz transforms by more general classes of kernel operators obeying a certain point separation criterion for their Fourier multiplier symbols. See also [18] and [16] for natural questions in this direction. Li [12] used a criterion similar to Uchiyama's, to show that it was also a sufficiently representative class to characterize BMO by means of commutators.

All of these results date back to the 70s, 80s and 90s and consider H^1 spaces in one parameter and simple, i.e. non-iterated commutators.

It is well known that the product theory and with it the product BMO space, as identified by Chang and Fefferman [4,3] have more complicated structure. We remind of Carleson's interesting example [2] illustrating this difference. The techniques to tackle the analogs of the above questions in several parameters are very different and have brought, with the works of Lacey and his collaborators, valuable new insight and use to existing theories, for example in the interpretation of Journé's lemma in combination with Carleson's example.

Ferguson and Lacey proved in [6] that the iterated commutator of the Hilbert transform and multiplication by a symbol b characterize BMO, and with it, they proved the equivalent weak factorization result for H^1 on the bidisk. Lacey and Terwilleger extended this result to an arbitrary number of iterates in [11], requiring thus, among others, a refinement of Pipher's iterated multi-parameter version of Journé's lemma. The real variable analog, the result of Coifman, Rochberg and Weiss [5] using Riesz transforms instead of Hilbert transforms, was extended to the multi parameter setting in [10]. In this current paper, we extend in part, the direction of Uchiyama and Li to several parameters. We formulate a sufficient condition on a family of Calderón–Zygmund operators, so that their iterated commutators characterize BMO:

For vectors $\vec{d} = (d_1, \dots, d_t) \in \mathbb{N}^t$, we consider product spaces

$$\mathbb{R}^{\vec{d}} = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t}.$$

For each $1 \leqslant s \leqslant t$, we have a collection of Calderón–Zygmund operators $\mathcal{T}_s = \{T_{s,1}, \dots, T_{s,n_s}\}$, whose kernels are homogeneous of degree $-d_s$, with Fourier multiplier symbols $\Theta_s = \{\theta_{s,k_s} \in \mathcal{C}^\infty(\mathcal{S}^{d_s-1})\colon 1 \leqslant k_s \leqslant n_s\}$ that are in turn homogeneous of