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Gaussian bounds for higher-order elliptic differential operators with Kato type potentials

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A B S T R A C T

Let $P(D)$ be a nonnegative homogeneous elliptic operator of order $2m$ with real constant coefficients on $\mathbb{R}^n$ and V be a suitable real measurable function. In this paper, we are mainly devoted to establish the Gaussian upper bound for Schrödinger type semigroup e^{-tH} generated by $H = P(D)+V$ with Kato type perturbing potential V, which naturally generalizes the classical result for Schrödinger semigroup $e^{-t(\Delta+V)}$ as $V \in K_2(\mathbb{R}^n)$, the famous Kato potential class. Our proof significantly depends on the analyticity of the free semigroup $e^{-tP(D)}$ on $L^1(\mathbb{R}^n)$. As a consequence of the Gaussian upper bound, the L^p-spectral independence of H is concluded.

1. Introduction

Let $m \geqslant 1$, $P(D)$ be a nonnegative homogeneous elliptic operator of order $2m$ on $L^2(\mathbb{R}^n)$ with real constant coefficients, V be a measurable function on $\mathbb{R}^n$ and $V_\pm$ denote the positive and negative parts of V. If V_- satisfies the following small form perturbation:

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$$\int V_-|f|^2\,dx \leqslant \epsilon \int \bigl(P(D)f\bigr)\bar{f}\,dx + C_\epsilon \int |f|^2\,dx, \quad f \in C_c^\infty(\mathbb{R}^n), \tag{1.1}$$

for some $0 < \epsilon < 1$ and $C_\epsilon > 0$, then we can define a semibounded self-adjoint operator $H := P(D) + V$ by the following closed form sum

$$Q(f) := \int \bigl(P(D)f\bigr)\bar{f}\,dx + \int V|f|^2\,dx, \tag{1.2}$$

where $f \in W^{m,2}(\mathbb{R}^n)$ such that $\int V_+|f|^2\,dx < \infty$ ($W^{m,2}(\mathbb{R}^n)$ denotes the Sobolev space). Thus it follows that the self-adjoint operator H generates a semigroup e^{-tH} on $L^2(\mathbb{R}^n)$, which clearly admits a distributional kernel $K(t,x,y) \in \mathfrak{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ for each $t > 0$ by Schwartz kernel theorem (see e.g. Hörmander [21, Chapter 5]).

It is of great interest to see if the heat kernel $K(t,x,y)$ of the semigroup e^{-tH} has Gaussian point-wise estimates which are very useful to the spectral theory of H, Riesz transforms and Hardy space theory associated to H (for instance, see [1,5,8,13,15,17,25] and references therein for more interesting works). As concerned with $H = P(D) + V$, if $n < 2m$ and the condition (1.1) is satisfied, then there exist constants $C, c, \omega > 0$ such that the kernel $K(t,x,y)$ satisfies the following Gaussian estimate (e.g. see Davies [10], Barbatis and Davies [4])

$$\bigl|K(t,x,y)\bigr| \leqslant Ct^{-\frac{n}{2m}} \exp\biggl\{-c\frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + \omega t\biggr\}, \tag{1.3}$$

which obviously implies that the semigroup e^{-tH} on $L^2(\mathbb{R}^n)$ can be extended to a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ for all $1 \leqslant p < \infty$ and also bounded on $L^\infty(\mathbb{R}^n)$. In particular, one can choose $\omega = 0$ in (1.3) if $V \geqslant 0$. Moreover, when $n < 2m$, $H = (-\Delta)^m + V$ and $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the following sharp point-wise estimate holds (for instance, see [2–4,14]):

$$\bigl|K(t,x,y)\bigr| \leqslant C_r t^{-\frac{n}{2m}} \exp\biggl\{-d_m\frac{|x-y|^{2m/(2m-1)}}{rt^{1/(2m-1)}}\biggr\}, \tag{1.4}$$

where $C_r > 0$ for all $r > 1$ and

$$d_m = (2m-1)(2m)^{-\frac{2m}{2m-1}} \sin\frac{\pi}{4m-2},$$

which is also the sharp kernel estimates of the $e^{-t(-\Delta)^m}$ for $n < 2m$.

However, if $n > 2m$ and no other conditions besides (1.1) are imposed on V, then generally, this kind of Gaussian estimate (1.3) may fail, and the semigroup e^{-tH} has only a partial L^p-theory where p belongs to some bounded interval around 2 (for example, see [6,10,11,28,29]). More specifically, for a class of more general second order elliptic operators with the condition (1.1) (including Schrödinger operator $-\Delta + V$), Liskevich, Sobol and Vogt [29] proved that there exists a maximal bounded interval $(p_{\min}, p_{\max})$